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FOR STUDENTS OF AGRICULTURE

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MATHEMATICS

FOR

STUDENTS OF AGRICULTURE

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BY

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PREFACE

THIS book presents a year's work in mathematics for students taking agricultural courses in secondary, vocational, and technical schools and in colleges and universities. By omitting the more elementary or the advanced topics, as may seem desirable, or by properly selecting the exercises to be solved, the book will serve quite naturally for a half-year's work.

The modern agriculturalist must devote a larger and larger part of his time to mental activity along scientific lines. To do this rationally he must have better methods for, and more practice in, estimating, computing, and comparing the materials and processes with which he works. The object of the present text is to furnish a better basis for effecting these ends.

The text has been used in the form of mimeographed notes for several years in the classroom, and has been modified from time to time in the light of experience and of new conditions. It was soon found that it was not safe to assume too much preparation on the part of students taking the course. Consequently, material is presented in the text for study and practice in basic elementary processes in arithmetic, algebra, and geometry. Some study of this material will suffice for those students who have not had the proper preparation for the course as well as for those students who wish to review elementary facts and principles.

The unifying element in the book is the attempt to make the principles of arithmetic, algebra, geometry, trigonometry, and graphic representation function with the student's interest and point of view. It is hoped that in this way he may find a natural tendency to use this most basic science in the solution at least of those problems which arise in his own sphere of activity. We may, therefore, expect the student to acquire in this way not less of mathematical value and content but more of it.

The following are features of various chapters:

Chapter I presents suggestions in drawing as a means of reviewing facts from geometry and connecting them naturally with such observations as the student can make; for example, estimating and computing the grade of a roof or a hill or a ditch or a railroad, the

projections of the sides of a farm contour on a base line, scale drawings of farms and farm buildings, and the composition and resolution of forces graphically.

In Chapter II a basis is presented for rational calculation and for estimation, including the question of accuracy. The topics considered are percentage, profit and loss, bank discount, elementary mensuration of common plane figures and solids, including rules for estimating silos.

In Chapter III a treatment of indirect measurement and proportion is given. This includes also the study of the sine, cosine, and tangent ratios and tables for them, with applications to simple problems in farm surveying.

A review of the main facts of elementary algebra may be obtained from Chapter IV. Graphic representation of quantities, including price curves, weight and age curves, leveling, etc., is presented in Chapter V. A rather fundamental treatment of compound interest, annuities, depreciation, and farm loans is given in Chapter IX.

Chapter X treats averages and mixtures. It includes a discussion of dairy problems, arithmetic and geometric means, weighted averages, the median, the mode, fertilizer mixtures, and cement mixtures.

The chapter on Land Surveying (Chapter XIII) may be of interest from the standpoint of the applications and the new material.

Chapter XIV is entitled Simple Machines. It presents a treatment of levers, the wheel and axle, the inclined plane, the screw, pulleys, safe loads for beams, work, horse-power, working day for a horse, draft, etc.

The tables printed in the Appendix at the end of the book are adapted to the requirements of computing and estimating the exercises and examples. Throughout the text the exercises and illustrations are drawn largely from experience and the point of view of the agriculturalist.

The author desires to acknowledge his indebtedness to Dean Alfred Vivian and members of the faculty of the College of Agriculture of The Ohio State University, to R. D. Bohannan, Professor of Mathematics in The Ohio State University, and to E. R. Hedrick, Professor of Mathematics in the University of Missouri, for the inspiration of their interest and suggestions.

S. E. Rasor.

The Ohio State University.

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A LIST OF SIGNS AND SYMBOLS

$+$, read *plus*. $-$, read *minus*.

$\times$, or $\cdot$, read *times*.

$\div$, read *divided by*.

$=$, read *equals*, or *is equal to*.

$\neq$, read *is not equal to*.

$<$, read *is less than*.

$>$, read is *greater than*.

$\therefore$, read *therefore*.

$'$, read *feet*, e.g. $2'$, means 2 feet.

$''$, read *inches*, e.g. $2''$, means 2 inches.

( ), Parenthesis.
[ ], Brackets.
{ }, Braces.

These are signs of aggregation. The expressions which they inclose are to be treated together as one quantity.

$a : b$, read a *is to* b.

$\sqrt{a}$, read *square root of* a.

$\sqrt[n]{a}$, read n^{th} *root of* a.

a^n, read a *to the* n^{th} *power*, or a *exponent* n.

$\log_a n$, read *logarithm of* n *to the base* a.

$\perp$'r, read *perpendicular*, or *is perpendicular to*

$\angle$, read *angle*, $\measuredangle$, read *angles*.

$\triangle$, read *triangle*, $\triangle\!\!\triangle$, read *triangles*.

Rt. $\triangle$, read *right triangle*.

$\square$, read *parallelogram*.

MATHEMATICS FOR STUDENTS OF AGRICULTURE

CHAPTER I

INTRODUCTION — DRAWING — GRAPHIC SOLUTIONS

1. Instruments and Materials. A large number of the exercises and problems in this book are to be solved *graphically, i.e.* by drawing appropriate diagrams. To do this work neatly and accurately, a few simple drawing instruments are essential. The materials and instruments needed for the work are described briefly below.*

(*a*) *Pencils.* One medium hard pencil sharpened to a *round point* may be used for lettering. A hard pencil sharpened to a round point may be used for sketching and marking points on paper. Another hard pencil, sharpened to a *chisel-point* (Fig. 1), may be used to draw straight lines. A piece of sandpaper may be used to whet the points.

FIG. 1

(*b*) A *drawing board*, about 12 or 15 by 18 inches. It should be made of soft wood and have at least one straight edge.

(*c*) A *T-square*, 15 to 18 inches long. As shown in Fig. 2, a T-square consists simply of a wooden strip, *A*,

*Those students who are taking courses in drawing are already in possession of many of these instruments and materials needed for graphical work.

called the *head*, to which is fastened a thin, straight-edged *blade*, *B*.

(*d*) *Two transparent triangles.* Each of these should have one right angle. The acute angles of one triangle should be 45°, while the acute angles of the other should be 30° and 60° (Fig. 3).

T-SQUARE
FIG. 2

(*e*) A *scale* with the unit of length divided into tenths is to be preferred to one with the unit of length divided into eighths, twelfths, or sixteenths.

(*f*) A *protractor*, Fig. 3, 4 to 6 inches in diameter is recommended. It consists simply of angular graduations upon a semicircular piece of metal or some transparent material. It is used for laying off and measuring angles upon a drawing.

(*g*) *Two pairs of compasses.* One should be a pair of 6-inch pencil compasses, the other should be inking compasses of the same size.

(*h*) A pair of 6-inch *hairspring dividers.* These are to be used in calculating lengths, distances, etc. on drawings made to scale.

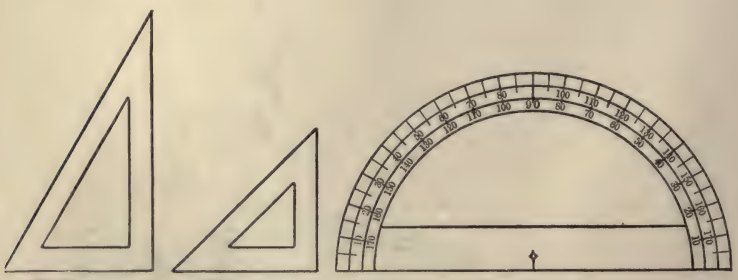

30° TRIANGLE 45° TRIANGLE PROTRACTOR

FIG. 3

2. General Directions. All work must be done with neatness and accuracy. The acquisition of these two virtues in graphical work may require an unusual amount of time and effort. However, when they are once established as habits, much time will be saved in avoiding and in detecting errors, as well as in checking results.

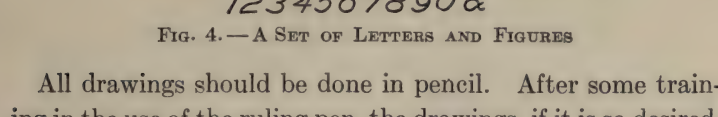


FIG. 4. — A SET OF LETTERS AND FIGURES

All drawings should be done in pencil. After some training in the use of the ruling pen, the drawings, if it is so desired, may be " inked in " with drawing ink.

Figure 4 shows a set of letters and figures. They may be used in *lettering* descriptive work on a drawing instead of using ordinary writing.

EXERCISES

1. Draw a straight line. For this purpose two things are required:

(*a*) A plane or flat surface on which to draw the line.

(*b*) A straightedge to be used as a guide.

2. State some facts about a straight line by testing a straightedge. To do this, select two points through which the straightedge is to pass, draw a line through them, reverse the straightedge, and redraw the line.

How many points determine a straight line? How many points may two distinct straight lines have in common? When does a straight line lie wholly in a plane?

3. On what surfaces, other than the plane, is it possible to lay a straight line so that each of its points coincides with the surface?

In this connection, examine a cylinder; a cone; a riding saddle; a capstan.

4. The part of a straight line between two of its points is called a ***line segment*** or simply a ***segment***, and is designated by its end points,

or sometimes by a small letter near the middle of the segment. Thus we speak of the "*segment AB*" or the "*segment a*." See Fig. 5.

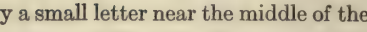


FIG. 5

On a given straight line construct a segment equal to a given segment. Do this in two ways.

(*a*) By the use of a pair of dividers.

(*b*) By the use of a scale or a straightedge, — perhaps an inch scale, or a centimeter scale.

Which of these methods is the more accurate? How could the accuracy of the second method be improved?

5. Locate any four points A, B, C, D in the plane and draw the straight lines which any pair of them determines. In general, how many of these lines are there? Under what conditions are there less than this number of lines?

6. Draw four straight lines at random. In general how many points of intersection are there? When are there less than this number of intersections? What are the names of some of the possible figures obtained in this way?

7. Given two unequal segments a and b. Draw the segment equal to $a+b$. Equal to $a-b$ when $a>b$. Draw the segment equal to $a-b$ when $a<b$.

8. Draw the shortest distance between two given points A and B in a plane.

9. Determine all the points which are the same distance from two given points. For example, how could a fence be located so that any place on it would be equidistant from a certain house and barn?

10. Determine the point C midway between two points A and B. Draw a perpendicular to C by using dividers and ruler. Find the mid-point C by using the draftsman's triangle alone.

11. Draw the shortest distance from a point A to a given straight line. From a point A to a given circle.

12. By use of ruler and compass, bisect a given angle. Draw angles of 90°, 45°, 135°, 22½°.

13. Solve the previous exercise by use of a carpenter's square alone.

14. Draw in a circle, by use of ruler and compass, an inscribed square. An inscribed octagon. An inscribed hexagon.

15. Draw, with ruler and compass, right triangles with acute angles 30°, 45°, 60°.

16. How are the *sides* related in each of the triangles in the preceding exercise?

Do these relations in any way describe or determine the *angles* of these triangles, *i.e.* if these *ratios of the sides* in a right triangle were given, would you then be able to discover what *angles* they determined?

17. Prove that the image of an object as seen in a plane reflecting surface appears to be as far back of the surface as the object is in front of it, using the law of reflection that the angle of incidence is equal to the angle of reflection.

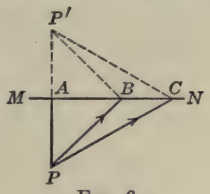


FIG. 6

NOTE. The *angle of incidence* is the angle which the path of a ray of light makes with the perpendicular to the reflecting surface as it strikes the surface. The *angle of reflection* is the angle the reflected ray makes with the perpendicular as it leaves the surface. In the accompanying figure i = the angle of incidence and r = the angle of reflection. Light, sound, and moving particles are reflected according to the law that $\angle i = \angle r$.

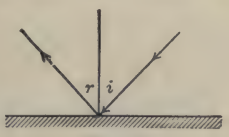


FIG. 7

18. Determine the path of a ray of light whose origin is at a given point A and which is reflected from a given straight line MN to a given point B.

19. A ladder leaning against a barn slips to the ground. In doing so, what curve does the center of the ladder describe if the foot of the ladder moves directly and horizontally away from the barn?

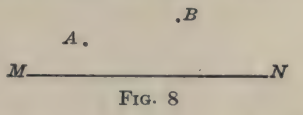


FIG. 8

[HINT. How far from the vertex of the right angle is the center of the hypotenuse in any right triangle?]

3. Drawing to Scale. Since it is not possible in most cases to draw objects full size on paper, the drawings are made *proportionally* smaller. This consists merely in making all the dimensions of the drawing a certain fraction of the true dimensions of the object. This is called drawing to

scale. An example is the plan of a house drawn to the scale: 1 inch (on the drawing) = 2 feet (on the house), *i.e.* to the scale of 1 in 24 or $\frac{1}{24}$.

A large number of problems may be solved graphically by drawing the given data to scale and then measuring the unknown parts. Rather accurate results are readily obtained in this way. It also furnishes a desirable, independent method of *checking* the calculations in problems for which an accurate solution has been obtained otherwise.

EXERCISES

1. Draw a ground plan of your home farm lot showing houses, barn, etc.; or such a plan of your city or suburban home.

2. Draw a ground plan, scale 1″ = 10′, of a barn 40′ × 50′ to house 4 horses and 10 cows, and with arrangements in it for tools and harness and extra pens for calves, etc.*

3. Three miles from a straight road is a fort with a gun whose range is five miles. Find, by drawing to the scale of 1″ to 2 miles, what distance along this road is commanded by the gun.

4. Find, by a scale drawing, the height of a tree which, at a distance of 350′ from an observer, subtends an angle of 22°.

FIG. 9

5. How can a carpenter divide a $4\frac{3}{4}$ in. board into 3 equal strips without the use of fractions? See Fig. 9.

6. Divide a given line segment into seven equal parts.

4. Projection of a Line Segment. To project a given segment, as AB, Fig. 10, upon a line PQ drop perpendiculars to PQ from the end points of the segment. Then CD is the projection of AB upon PQ and is called the ***horizontal***

* For suggestions on the architecture of farm buildings, see L. F. ALLEN, *Rural Architecture*, published by L. F. Allen, New York; EKBLAW, *Farm Structures*, Macmillan, 1916; G. G. HILL, *Practical Suggestions for Farm Buildings*, Wash., Gov. Ptg. Office, 1901; I. P. ROBERTS, *The Farmstead*, Macmillan, 1900.

projection in case PQ is horizontal. Projection on a vertical line is called ***vertical projection*** as BE, Fig. 10, in which AE is parallel to PQ, and BD perpendicular to it.

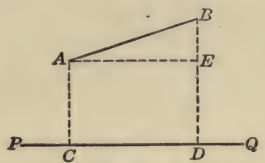


FIG. 10

Similarly, PQ may be projected upon AB (produced if necessary).

In the case of a hill, or a grade, or a rafter, the horizontal projection is called the ***run*** of the rafter, etc., and the vertical projection is called the ***rise***. The rise divided by run is called the ***slope;*** it is a measure of the steepness of the hill, or grade, or rafter. For example, if a rafter on a building

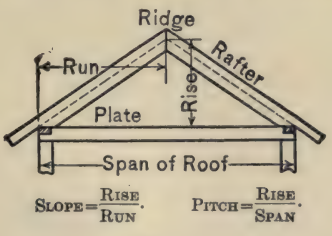


FIG. 11

24′ wide rises 6′ at the comb of the roof, we say it rises 6′ in 12′ or 1 in 2, *i.e.* the slope is $\frac{1}{2}$. If a hill rises 30′ in a run of 100′, it has a slope of 30 in 100, *i.e.* the grade is 30/100 or 30%. *A 1% grade,* as for a steam railroad, *rises 1′ in 100′.*

With reference to a roof, the word ***pitch*** is widely used; it denotes rise of the roof divided by the span or whole width of the building (Fig. 11). Notice that the pitch of a roof is half of the slope.

EXERCISES

Draw each of the following figures on an enlarged scale and project each segment upon the other. Specify the projection in each case. Find the horizontal and the vertical projections, and find, *by measuring*, the per cent grade for each segment.

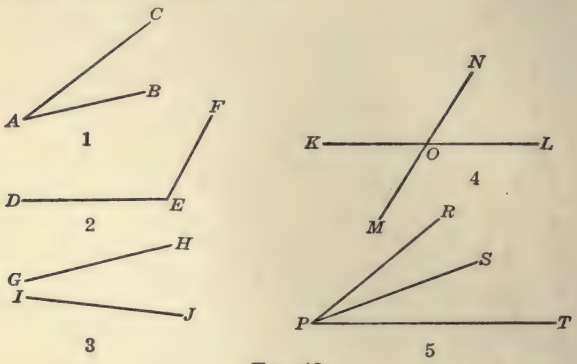


FIG. 12

6. The altitude of an equilateral triangle whose side is 12′ is projected upon one of the adjacent sides. Find the projection.

7. Find, by drawing to scale, the side of a square whose diagonal is 10 rd.

8. Show how to bisect a given angle by using two carpenter's squares.

[HINT. Use the geometric theorem that any point on the bisector of an angle is equidistant from the sides.]

9. Draw any oblique triangle. Draw the projections of two sides on the third side.

5. Graphic Resolution of Forces. A force may be represented graphically by a straight line whose *length* represents the *magnitude* of the force, and whose *direction* is that of the force. See § 213.

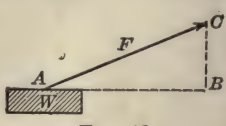


FIG. 13

In Fig. 13, F represents a force acting on W in the direction shown and at an angle BAC with the horizontal.

Project F on a horizontal line through the point where the force is applied to W. In this way the horizontal projection

AB is said to be the ***horizontal component*** of the force F and the perpendicular BC is called the ***vertical component*** of the force. F is thus *resolved* into vertical and horizontal components.

EXERCISES

1. A force of 10 pounds acts on a weight W at an angle of 30° with the horizontal. Draw the figure, making F 10 units in length. Project F on the horizontal line through the point of application of the force.

By the scale, determine the magnitude of each component in pounds. What is thus the amount and the direction of the horizontal and the vertical pull on W?

2. In drawing a sled a horse pulls 200 pounds on his traces, which make an angle of 10° with the horizontal. Plot to scale the figure representing these conditions and determine in this way what horizontal force is employed in moving the sled along the ground and also what force tends to lift the sled vertically.

3. In moving a weight on a sled as in the previous exercise, it was found that the vertical pull was 40 pounds. Determine the pull on the traces by drawing to scale. What is the horizontal component which moves the sled?

What is the effect of a "long hitch" in a case of this kind?

4. A weight of 40 lb. slides without friction down an incline which makes an angle of 30° with the horizontal. What force acting directly up the plane will keep the weight at rest?

[HINT. Draw as in the accompanying figure the segment AB 40 units in length and directly downward to represent the force due to the weight. Project this segment on the incline and call this projection AC. This is the component of the force down the plane. Now $\angle P = \angle B = 30°$ since $\triangle ABC$, PQR are similar. But ABC is half of an equilateral triangle and the side AC is one half of AB. Thus AC represents a force of 20 lb. Therefore a force of 20 lb. acting directly *up* the plane will keep the weight at rest. The pressure *on* the incline is represented by CB perpendicular to it. Find it.]

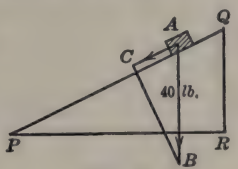


FIG. 14

5. What force is necessary to roll a barrel of cider weighing 200 lb. up an incline 12 ft. long into a wagon bed 3 ft. above the ground?

CHAPTER II

COMPUTATION — MEASUREMENT

6. Measurements Are Approximations. Numbers met with in actual life are in general approximations. For example, if we should measure the width of a room and find the measurement to be $13'\ 6\frac{5}{8}''$, obviously we could not be certain that this is the *exact* width. A more exact measurement could be obtained by using a ruler graduated to smaller divisions, for example to sixty-fourths of an inch. The width of the room is thus not measurable to an exact fraction of an inch. How nearly the measured width approximates the true width depends upon the wishes of the operator, upon his skill, upon the kind of instrument used, etc. Even the price of butter at 67¢ per pound, while seemingly exact, is only approximate, if the weighing process be considered.

Again, common fractions whose denominators contain factors other than 2 or 5 cannot be expressed exactly as terminating decimals. For example, one sixth of a dollar is $16.66\cdots¢ = 17¢$. The value may be expressed decimally to any degree of accuracy desired. Roots of numbers, while exact in themselves, are in general only approximations when expressed in decimal form.

Thus, $1/6 = 0.17$, or $= 0.167$, or 0.1667, etc., and $\sqrt{2} = 1.4$, or $= 1.414$, or $= 1.4142$, depending upon the degree of accuracy required. In all cases the number should express only the degree of accuracy demanded by the nature of the problem. The cost of food to \$0.001, or the volume of a barn to 1 cu. ft., are meaningless.

7. Accuracy of Processes Involving Approximations. The accuracy or precision of a measurement is indicated by *the number of significant* figures* used to express the result and not by the number of decimal places. For example, 52.3 inches = 4.83 feet = 1.35 meters = .293 rod according to the unit used. Notice that these four members represent the same length and are about equally accurate. They each contain three significant figures but in one case one decimal place is used, in another three decimal places. These numbers, and others like them, are said to be of *three-figure accuracy*. Such numbers may be obtained in measurements made with ordinary instruments and skill.

Four-figure accuracy is sufficient for a great many engineering operations. *Five-figure accuracy* results only from good instruments and great care. Thus a lot on Broadway, New York City, is measured to thousandths of a foot. Only experts can obtain six-figure accuracy, while *eight-figure accuracy* is as yet obtainable only in rare instances.

In working with approximate numbers, the accuracy of calculated results can seldom equal that of the original numbers used. If a measured quantity is expressed in three figures, the result of operating with it should be expressed in general in not more than three figures by merely dropping unnecessary decimals. For example, 2.36 feet = 2.36×12 inches = 28.3 inches, *not* 28.32 inches. Here 2.36 feet as a measurement means that the true result is somewhere between 2.355 and 2.365. This is written $(2.36 \pm .005)$ feet.

* All figures, other than zero, are significant in a number. A zero is significant unless used merely to locate the decimal point. In 206, 2.06, 0.206 the central zero is significant and these are called three-figured numbers. In 0.0003 the zeros are *not* significant and this is called a one-figured number. A final zero may be significant; thus 2.60 ft. means that the length measured is nearer 2.60 ft. than to 2.59 ft. or to 2.61 ft. If the result were written 2.6 ft., it would mean that the measurement had been made only to the nearest tenth of a foot.

Now $(2.36 \pm 0.005)12$ inches $= (28.32 \pm 0.06)$ inches, which shows that the final figure is in doubt. It should therefore be written 28.3 inches if no accompanying statement is made as to the error involved.

However, it is usual to determine the doubtful place in a result from accompanying statements of probable error in the numbers used, and then drop unnecessary figures beyond the first doubtful one.

8. Percentage. The word per cent is a contraction of the Latin word *percentum* and means by the hundred, *i.e.* hundredths. It is written %. It furnishes a convenient way of making comparisons between quantities, rates, etc., by using a hundred as the basis. Thus,

$$6\% \text{ means } .06 \text{ or } \tfrac{6}{100}$$
$$135\% \text{ means } 1.35 \text{ or } \tfrac{135}{100}$$

EXAMPLE. An analysis of a pound of alfalfa seed showed weed seed, $2\frac{9}{16}$ oz.; dirt, $\frac{1}{16}$ oz.; good seed, $12\frac{1}{2}$ oz.

The relative amounts of weed seed, dirt, and good seed may be more easily recognized by stating them in hundredths of a pound, *i.e.* in per cents. The analysis thus shows

weed seed $2\frac{9}{16}$ oz. $= .16$ lb. $= 16\%$
dirt $\frac{1}{16}$ oz. $= .06$ lb. $= 6\%$
good seed $12\frac{1}{2}$ oz. $= .78$ lb. $= 78\%$

Computations involving per cent contain the following three elements.

The ***Rate per cent***, which is the number of hundredths taken.

The ***Base***, which is the number on which the hundredths are computed.

The ***Percentage***, which is the result of taking a certain per cent of a number.

RULES. *Base times Rate = Percentage.*
Percentage divided by Base = Rate.
Percentage divided by Rate = Base.

EXAMPLE 1. Find 8% of \$150.

8% of \$150 = .08 times \$150 = \$12.00

EXAMPLE 2. What per cent of 750 is 25?

25 is $\frac{25}{750}$ of 750, but $\frac{25}{750} = \frac{1}{30} = .03\frac{1}{3} = 3\frac{1}{3}\%$.

EXAMPLE 3. \$1.05 is 15% of what amount?

FIRST SOLUTION: By the rule, the amount required or the base = \$1.05 ÷ .15 = \$7.00

SECOND SOLUTION: If 15% of some number = \$1.05, then 1% of that number = \$.07, and 100%, or the number, = \$7.00

THIRD SOLUTION: Let x be the amount required. Then

$$.15\,x = \$1.05,$$

whence

$$x = \frac{\$1.05}{.15} = \$7.00$$

The following equivalents may be found useful in estimating and in computing:

$6\% = .06 = \frac{3}{50}$.	$12\frac{1}{2}\% = \frac{1}{8}$.	$62\frac{1}{2}\% = \frac{5}{8}$.
$6\frac{1}{4}\% = .06\frac{1}{4} = \frac{1}{16}$.	$16\frac{2}{3}\% = \frac{1}{6}$.	$66\frac{2}{3}\% = \frac{2}{3}$.
$6\frac{2}{3}\% = \frac{1}{15}$.	$25\% = \frac{1}{4}$.	$83\frac{1}{3}\% = \frac{5}{6}$.
$7\frac{1}{7}\% = \frac{1}{14}$.	$33\frac{1}{3}\% = \frac{1}{3}$.	$87\frac{1}{2}\% = \frac{7}{8}$.
$8\frac{1}{3}\% = \frac{1}{12}$.	$37\frac{1}{2}\% = \frac{3}{8}$.	

EXERCISES

1. Find $6\frac{2}{3}\%$ of 300. Of 75. Of 18. Find 23% of 1684. $\frac{7}{8}\%$ of \$24. $3\frac{1}{4}\%$ of 180 lb.

2. What per cent of 15 is 8? What per cent of 20 is 9? What per cent of $12\frac{1}{2}$ is 3? What per cent of $3\frac{4}{5}$ is $2\frac{2}{5}$? What per cent of 8 is 10?

3. What per cent of $37\frac{1}{2}$ feet is 5 feet? What per cent of 7 yards is 7 feet? What per cent of 2 miles is 528 feet? What per cent pure is an 18-karat gold ring?

4. 6 is 8% of what number? 4 is 10% of what? 18 is $2\frac{1}{2}\%$ of what? 24 is 12% of what? 216 is 13% of what? 6 feet 2.8 inches is 22% of what length? 9.9 inches is $12\frac{1}{2}\%$ of how many yards?

5. In a certain fertilizer for corn land, $47\frac{1}{2}\%$ is cotton-seed meal, $43\frac{3}{4}\%$ is phosphoric acid (phosphorus), and the rest kainit. How many pounds of each are there in a ton of the fertilizer?

How much acid phosphate containing 16% phosphoric acid is required to supply the phosphoric acid?

How much potash is there in the kainit required if kainit contains 13.54% potash?

6. Cooked eggs contain 12% protein and cooked roast beef contains 22.4% protein. If one egg weighs 2 oz., how many eggs will furnish as much protein as 1 pound of roast beef? Which is the more economical food with eggs at 65¢ per dozen and beef at 25¢ per pound?

7. Ear corn shrinks in weight about 3% per month in drying. On Oct. 1 a man has 20 tons of corn. What should it weigh the following March 15? Which is the better proposition, to sell it "green" at $30 per ton or hold it till March 15 and sell it for $35 per ton, allowing 6% interest on the value of the corn? What "present" selling price (October 1) at 6% interest would be the equivalent of $35 per ton after drying (March 15)?

8. Workmen strike for a 10% raise on their present scale of $2.25 per day. If they "stay out" 50 days, how long will it take them under the new scale to regain their loss, if they work 285 days to the year, not considering interest on the money involved?

9. The champion Jersey cow (1914) (Sophie 19th of Hood Farm) produced 999.14 pounds of butter fat in 365 days. Her milk tested 5.69% butter fat.

The champion Guernsey cow of the world, Murne Cowan (for a time champion of all breeds, 1915), produced 24,008 pounds of milk in 365 days, testing 4.57% butter fat.

When Finderne Pride Johanna Rue (Holstein) completed her record in June, 1915, she became the world's champion dairy cow. She produced, in 365 days, 28,403.7 pounds of milk containing 1176.47 pounds of butter fat.

Compare these three breeds of dairy cows on the basis of the amount of milk produced; on the basis of the amount of butter fat produced; on the basis of the percentage of butter fat in the milk.

9. Trade Discounts. Profit and Loss. Merchants, jobbers, manufacturers, publishers, etc. usually have a fixed price, called the *list price*, from which they allow *discounts* to the trade, according to the credit of the customer, the amount purchased, the time of payment, etc. If there is more than one discount, it is understood that the first is the

discount from the list price, the second is the discount from the *remainder*, etc.

The per cent of profit or loss in a transaction is usually reckoned on the cost. For example, if an article costing \$6 is sold for \$7.20, the gain is \$1.20, *i.e.* it is $\frac{1}{5}$ or 20% of the cost.

EXERCISES

1. What is the cash cost of 12 bales of barb wire at \$3.75, subject to a 10% discount, and a further discount of 2% if paid in 10 days?

2. Supplies for fences amounted to \$58.75. If a discount of $12\frac{1}{2}$% is allowed with an additional 2% off for cash, what is the cash cost of the supplies?

3. A dealer is offered 40% and 15% off on a bill of \$126.83. What is the net price?

4. Which is the better proposition, to pay a bill according to a straight discount of 25%, or according to successive discounts of 15%, 5%, 5%? How much better is it?

5. A dealer buys apples at \$6.75 per barrel (3 bu.) and retails them at 75¢ per peck. What is his gain or loss on 30 barrels, allowing 10% loss by decay?

6. A renter raises 60 bu. per acre on 40 acres of corn where land is valued at \$180 per acre. He sells the corn at \$1.25 per bushel, pays a rental of 8% of the farm value, and estimates the cost of crop production at \$45.75 per acre. How much and what per cent profit does he make on the cost of production?

7. A quantity of wheat was sold in succession by three dealers, each of whom made a profit of 4%. The last dealer received \$1384. What did it cost the first dealer?

8. What per cent profit does a grocer make on sugar bought at $14\frac{1}{2}$¢ per pound, depreciated 10% from drying and downweight, and sold at 16¢ per pound?

9. A man buys 2000 bu. of potatoes at \$1.45, sells $\frac{3}{4}$ of them at 20% gain and the remainder at a loss of $11\frac{1}{9}$%. Find his net rate of gain.

10. Goods are bought at 10% below list price and sold at 20% above list price. What is the per cent profit?

11. If a dealer gets discounts of 40% and 15%, what discount can he give on the list price to make 30% on his investment?

10. Simple Interest. *Interest* is money paid for the use of money. The *principal* is the amount borrowed. The *rate* is the part of the principal to be paid for its use, usually for one year, even though it may be collected semiannually or quarterly if so specified. Thus the statement that a note or a bond is issued at 6% interest payable semiannually means that interest is payable semiannually at the rate of 6% per year. In computing simple interest, thirty days are considered an interest month and 360 days an interest year. The United States Government and some banks and business houses use the exact interest method. This method takes account of the *exact number of days in the time interval*, and proceeds upon the basis of 365 *days to the year*. Banks usually compute interest for the exact number of days.

11. Partial Payments. Sometimes it is convenient to pay part of a note before it matures. Such payments should be *indorsed* on the back of the note, with the *date* and the *amount* of the payment. They are known as partial payments.

The Supreme Court of the United States, and nearly all of the states, use the following rule for partial payments, called the *United States Rule.*

1. *Find the amount of the principal to the time of the first payment; if the payment equals or exceeds the interest, subtract the payment from the amount and treat the remainder as a new principal.*

2. *If the payment is less than the interest, find the amount of the same principal to the time when the sum of the payments shall equal or exceed the interest due, and subtract the sum of the payments from the amount.*

3. *Proceed in the same manner with the remaining payments until the time of settlement.*

EXERCISES

1. \$304.75 Chicago, Ill., March 10, 1912.

One year after date, for value received, I promise to pay to G. Botha, or order, Three Hundred Four and $\frac{75}{100}$ Dollars, with interest at 6%.

B. Thoba.

Indorsements: July 12, 1912, \$100; Nov. 30, 1912, \$100.

What was due June 25, 1914?

2. \$429.30 Columbus, O., April 13, 1913.

On demand, I promise to pay Spencer and Arps, or order, Four Hundred Twenty-nine and $\frac{30}{100}$ Dollars, value received, with interest at 6%.

R. Macklin.

Indorsements: Oct. 2, 1913, \$10; Dec. 14, 1914, \$120; July 1, 1915, \$25.

What was due Jan. 1, 1916?

3. A note for \$650, at 5%, is dated June 1, 1907, and has the following partial payments indorsed on it: Aug. 2, 1907, by labor, \$5.00; Aug. 8, 1907, by labor, \$2.00; Oct. 1, 1907, \$95; June 1, 1908, \$275; Dec. 1, 1908, \$140. How much is due Jan. 1, 1909?

12. Present Worth. Bank Discount. The *present worth* of k dollars, payable at a given future time, is the sum that will amount to k dollars at the end of the time, at a specified rate of interest.

The interest on the present worth of a debt due at a future time is called true discount. It is the difference between the debt and its present worth.

Example. What is the present worth of \$318 due in 1 year if money is worth 6%?

Solution: If x is the present worth, then the amount of x for 1 yr. at 6% is equal to \$318, *i.e.*

$$x(1.06) = \$318,$$

$$x = \frac{\$318}{\$1.06} = \$300.$$

In any case, we have the following rule:

Rule for Present Worth. *To find the present worth, divide the given sum by the amount of* \$1.00 *for the given time at the given rate.*

13. Bank Discount. When a loan is made at a bank, *the interest is paid in advance,* and the interest thus paid is called ***bank discount.***

The holder of a note which is due at a certain time may want the value of the note before it is due. In this case he usually takes it to a bank where the note is ***discounted***, *i.e. the interest on the full value of the note at maturity, from the time of discount to the time of maturity, is subtracted from the amount of the note, and the remainder, called the* ***proceeds****, is paid to the holder of the note.* The note then becomes the property of the bank.

EXAMPLE. A note of \$600, dated March 1, 1908, interest 6%, to run 4 months, was discounted at 6%, March 20, 1908. Find the proceeds.

Amount of \$600 for 4 mo. @ 6%	\$612.00
Interest on \$612 @ 6% for 105 days, March 20 to July 3, (called the bank discount)	10.71
Proceeds, after deducting the bank discount,	\$601.29

EXERCISES

1. What is the present worth of \$460.50 due in 3 yr. 9 mo. 18 days, if money is worth 6%? *Ans.* \$375.

2. Find the present worth of \$375 due in 3 yr. 3 mo. at 6%. *Ans.* \$313.81

3. Which is worth most, \$320 in 12 mo., \$310 in 6 mo., or \$300 cash, if money is worth 6%? *Ans.* The first.

4. A note of \$700 dated Aug. 9, 1915, is due Feb. 12, 1916, and bears 6% interest. What is the present worth of this note on Oct. 8, 1915? What are the proceeds on that date if discounted in a bank at 7%?

5. A man buys a farm on March 25, 1911, agreeing to pay \$2700, Oct. 1, 1911, and \$2500, Jan. 1, 1912. If a discount of 7% is allowed for cash, how much will he gain by borrowing the money on his own note discounted at 6%? How much must he borrow?

6. A note of \$560 to run 3 months, dated Aug. 4, 1914, bearing 5% interest, was discounted at 6% on Sept. 19. Find the proceeds of this note.

7. A man sold his farm for \$6350, taking in payment a note at 5% interest due in 6 mo. He at once sold the note to a bank which discounted it at 6%. What were the proceeds, *i.e.* what did he get in cash for his farm?

8. A note of \$350 due in one year at 6% was discounted 54 days before it was due at 6%. What were the proceeds of the note?

14. Elementary Mensuration. Angles. In the following pages we shall treat angles, the lengths of lines and their ratios, the areas of surfaces, and the volumes of solids.

If two straight lines proceed from the same point they form a ***plane angle***. This point is the ***vertex*** of the angle. If there is only one angle at a vertex, the angle may be denoted by a capital letter at the vertex, as $\angle O$, Fig. 15. Any angle may be denoted by three letters, as $\angle AOP$, Fig. 15, the middle letter being the vertex.

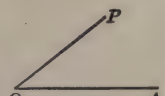


Fig. 15

An angle may be thought of as generated by a line turning about its end-point as a pivot. Thus, if the ray OP, Fig. 16, revolves about O, the amount of the turn from OX to OP measures the angle XOP indicated by the arrow.

If the ray continues to rotate counterclockwise until it is exactly *opposite* its original position it generates a ***straight angle***, as XOX', Fig. 16. If the ray continues till it reaches its original position, it generates a complete rotation, or one ***revolution***.

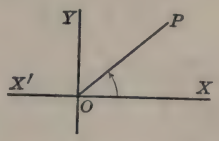


Fig. 16

A unit of measure for angles is the ***degree***. A complete rotation is 360 degrees, written 360°. Let the student state the definitions of *right angle*, *acute angle*, *obtuse angle*.

15. Triangles. Let the student give definitions of *triangle, equilateral triangle, isosceles triangle, right triangle* (Fig. 17).

The sum of the three angles of any triangle is two right angles, or 180°.

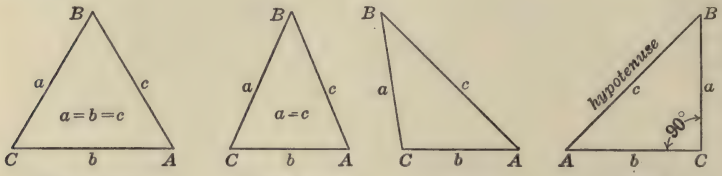


FIG. 17

It is customary, as shown in Fig. 17, to designate the *sides* of a triangle by the three small letters a, b, c and the *angles opposite* these sides by the corresponding capital letters, A, B, C, respectively.

In a right triangle the square of the hypotenuse equals the sum of the squares of the other two sides, i.e. from Fig. 18,

$$c^2 = a^2 + b^2.$$

This fact is called the *Pythagoras theorem.**

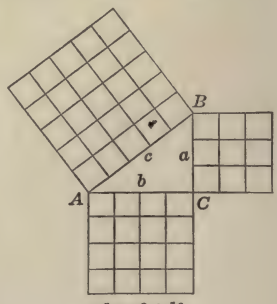


FIG. 18

When the base and the altitude of a triangle are given the area is

$$A = \frac{1}{2}\text{base} \times \text{altitude}.$$

When the three sides are given the area is

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

* Tables of squares and square roots are found in the Appendix. They may prove of assistance in the computation.

where a, b, and c denote the three sides, respectively, and

$$s=\frac{1}{2}(a+b+c).$$

State in words the formula for the area of a triangle when the three sides are given.

EXAMPLE. What is the area of a triangle whose sides are 15.3, 16.4, and 17.7 rods.

SOLUTION: Let $a=15.3$, $b=16.4$, $c=17.7$; then

$$s=\tfrac{1}{2}(a+b+c)=24.7,\ s-a=9.4,\ s-b=8.3,\ s-c=7.0$$

Therefore $A=\sqrt{(24.7)(9.4)(8.3)(7.0)}=116.2$ square rods.

EXERCISES

1. A ladder 16 ft. long and standing on level ground just reaches a window 12 ft. from the ground. Assuming the wall to be perpendicular, how far from the base of the wall is the foot of the ladder?

2. Find the length of the hypotenuse of a right triangle whose other two sides are equal and whose area is two acres.

How many degrees are there in each of the acute angles of this triangle?

3. Find the area of a triangle whose sides are 13, 18, and 21 in.

4. Find the altitude and the area of an isosceles triangle whose equal sides are 8 ft. long, and whose base is 6 ft. long.

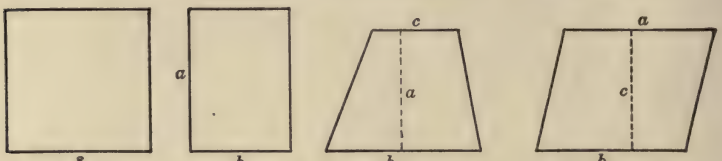
5. Find the length of the longest umbrella that can be put diagonally in a trunk which measures $20\times 20\times 34$ inches.

6. The dimensions of a barn are $20'\times 44'\times 48'$. What is the length of a wire which connects the upper corner at the eaves to the opposite lower corner?

7. A house is located 10 rods from a certain river, and a barn is 15 rods from this house and 19 rods from the same river. Find the point on the river equidistant from the house and the barn and find this distance.

8. A farmer wishes to run a pipe from some point on the river of Ex. 7 to the house and another pipe from the same point on the river to the barn. Where should it be laid to require the least length of pipe? Find this length.

16. Quadrilaterals. Let the student define *quadrilateral, square, rectangle, parallelogram, trapezoid.* (See Fig. 19.)



SQUARE RECTANGLE TRAPEZOID PARALLELOGRAM

FIG. 19

The areas of such quadrilaterals may be found by the following rules.

1. SQUARE:

$$Area = s^2,$$

where s = the length of one side.

2. RECTANGLE OR PARALLELOGRAM:

$$Area = base \times altitude.$$

3. TRAPEZOID:

$$Area = altitude \times half\ the\ sum\ of\ the\ parallel\ sides,\ i.e.$$

$$= a\frac{(b+c)}{2},$$

where a is the altitude, and b and c are the parallel sides.

EXERCISES

1. State in words the rule for the area of a trapezoid.

2. Find the area of a trapezoidal field whose parallel sides are 10 rods and 16 rods long and the distance between them is 8 rods.

3. Figure 20 as dimensioned represents the gable end of a building. Compute the length of the rafter OR and the area of $ABDOC$. What is the rise of the rafter per foot of its run? What is the pitch of the roof? (See § 4.)

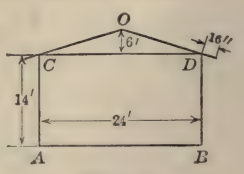


FIG. 20

4. Show how a carpenter "lays off" a rafter by use of a steel square, *i.e.* how he cuts the desired bevel on the rafter, making use of the ratio of rise to run.

5. Compute the pitch and the length of each rafter and the area of the entire gable end of the building as shown in Fig. 21.

6. ABC is a triangle, two of whose sides are $AC = 17'$ and $BC = 10'$. The projection of BC on the base AB is $6'$. Draw the several cases possible here and find the area of ABC for each case.

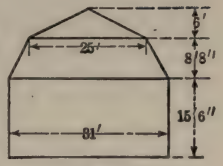


FIG. 21

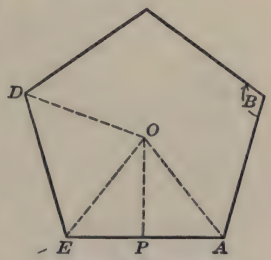
17. Polygons. Any plane figure bounded by straight lines is called a ***polygon***. The sum of its sides is called the ***perimeter*** of the polygon.

A polygon of five sides is called a ***pentagon***; one of six sides, a ***hexagon***; one of eight sides, an ***octagon***; one of ten sides, a ***decagon***, etc.

A ***regular polygon*** is one whose sides are all equal, and whose angles, between adjacent sides, are all equal.

The ***center of a regular polygon*** is the common center of its inscribed and circumscribed circles (§ 19).

A regular polygon can be divided into as many equal triangles as the polygon has sides. Each such triangle has one vertex at the center of the polygon, and its altitude, called the ***apothem*** of the polygon, is the distance from the center of the polygon to the center of a side, as OP, Fig. 22.



REGULAR POLYGONS

FIG. 22

An ***interior angle*** of a regular polygon, as B, Fig. 22, is equal to $(n-2)180° \div n$ where n is the number of sides of the polygon. Thus, for the regular pentagon, $n = 5$, and hence $B = (5-2)180° \div 5 = 108°$.

The area of a regular polygon is

$$A = \frac{1}{2}\textit{apothem} \times \textit{the perimeter}.$$

EXERCISES

1. Find the area of a regular hexagon whose side is 1″.

2. The side of a regular decagon is 1″ and its apothem is 1.54″ Find its area.

3. Prove the statement in the text that the interior angle of a regular polygon is equal to $(n-2)180° \div n$, by drawing the equal triangles *AOE*, *EOD*, etc. as in Fig. 22, and taking the sum of their interior angles less the angles at the center.

4. Find the number of degrees in an interior angle of a regular hexagon. A regular octagon. A regular decagon. A regular 48-gon. A regular 500-gon.

18. Irregular Areas. I. Straight Boundaries. If a tract is bounded by straight lines, *i.e.* if its perimeter is a polygon (Figs. 24–26), the area may be computed by projecting the sides of the polygon upon some convenient base line. The resulting trapezoids may be combined to obtain the required area.

The area can also be computed by dividing the tract into triangles and measuring a sufficient number of dimensions of each triangle.

II. Curved Boundaries. A curved boundary occurs when a tract touches a stream or a curved road. Stakes are set at points along the curved part, so that it will approximate closely to straight lines between the stakes. The perimeter then takes the form of an irregular polygon and may be treated accordingly.

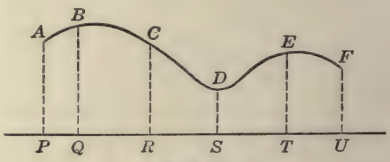


Fig. 23

For example, in Fig. 23, stakes are set at *A*, *B*, *C*, etc., along the boundary. The sides *AB*, *BC*, etc. are now projected on a convenient base line, and the **offsets** from the stakes to the base line are measured. The areas of the resulting trapezoids are then computed.

When the curved boundary is not too abrupt in any place, stakes may be set at *regular intervals* on the base line, and offsets to the curved boundary measured. The computation leads to the following general rule for offsets at regular intervals on the base line.

RULE. *Add all the offsets and subtract from this sum half the sum of the extreme offsets; multiply the remainder by the common distance between offsets.*

These methods form the basis for calculating areas in practical surveying. The base lines in surveying are usually the magnetic north-south lines.

EXERCISES

1. Find the area of a triangular field ABC in which $AE=5$ rods; $EF=9$ rods; $CE=10$ rods; $BF=7$ rods. Fig. 24.

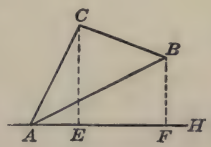


FIG. 24

[HINT. AH is the base line and AE and EF are the projections of AC and CB respectively upon it. The area of ABC is equal to the right triangle AEC plus the trapezoid $CEFB$ minus the right triangle AFB.]

2. Find the area of $ABCD$ in the diagram, taking

$PQ=20$ rods $\quad PD=40$ rods $\quad QR=10$ rods
$QC=50$ rods $\quad RS=25$ rods $\quad RA=15$ rods
$SB=45$ rods

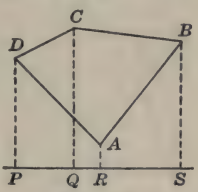


FIG. 25

[HINT. The area of $ABCD=PQCD+CQSB-DPRA-ARSB$, Fig. 25.

The area may also be found as follows: Draw a line through C parallel to PS and extend PD and BS to meet it. Take the area of the outside rectangle and subtract from it the areas of the other outside figures.]

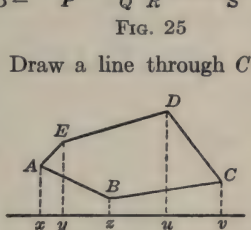


FIG. 26

3. Find the area of the five-sided field $ABCDE$ shown in Fig. 26, if

$xy=5$ rods	$Ax=15$ rods	$yz=10$ rods
$Ey=18$ rods	$zu=15$ rods	$Bz=4$ rods
$uv=12$ rods	$Du=22$ rods	$Cv=11$ rods

4. In surveying a field between the road AG and the river (Fig. 27), the surveyor measured along AG and drove a stake every 100 feet, at B, C, etc. At right angles to the road, the distances to the river were measured as follows: From B, 100 ft.; from C, 220 ft.; from D, 260 ft.; from E, 250 ft.; from F, 230 ft. Compute the approximate area of the field and its value at \$190 per acre.

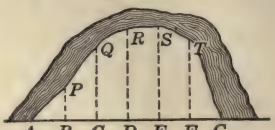


FIG. 27

5. Determine the area between a base line and an irregular boundary from the following table of offsets all on the *same* side of the base line. The distances are measured in feet between successive stakes on the base line. Give the result in square feet, and in acres.

Distances. . .	0	202	156	467	159	239
Offsets. . . .	20.7	31.5	42.6	53.2	36.1	40.7

Ans. 49,698 sq. ft.

6. The following offsets were taken on the same side of a line at distances of 100 ft. apart. Find the areas between extreme boundaries and the base line.

Distances .	0	100	200	300	400	500	600	700
Offsets . .	55.1	76.1	83.3	79.9	69.7	59.5	83.3	70.5

19. Circles. The circle occurs very frequently in applied mathematics, perhaps oftener than any other geometric form. Let the student state the definitions of *circle*, *center*, *radius*, *diameter*, *chord*, *arc*.

The area bounded by a chord and its arc is called a ***segment***, as AsB, Fig. 28. The area bounded by two radii and their intercepted arc is a ***sector***, as $OAsBO$, Fig. 28.

A straight line which touches a circle at only one point is a ***tangent***, as TPS, Fig. 28. A straight line which intersects a circle in two points is a ***secant***, as CF, Fig. 28.

A radius is perpendicular to the tangent at the point of contact.

A polygon is ***inscribed*** in a circle when its sides are chords of the circle, and the circle is then ***circumscribed*** about the

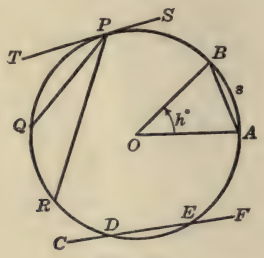


FIG. 28

polygon. A circle is ***inscribed*** in a polygon when each of the sides of the polygon is tangent to the circle.

20. Angles of a Circle. An angle formed by two radii is a ***central angle***, as AOB, Fig. 28.

An angle whose vertex lies on the circumference of a circle is an ***inscribed angle***, as QPR, Fig. 28.

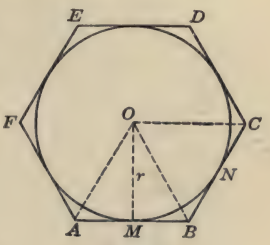


FIG. 29

I. *A central angle is said to be measured by the arc which it intercepts.* This means that a given central angle contains a number of unit angles equal to the number of unit arcs in the intercepted arc.

II. *An inscribed angle is measured by one half of its intercepted arc.* Thus, $\angle QPR$ = half the central angle whose arc is QR, Fig. 28.

III. *An angle between a tangent and a chord is measured by one half of the intercepted arc.* Thus, $\angle QPT$ = half arc PQ, Fig. 28.

21. Relations between Radius, Arc, Circumference of a Circle. In any circle, *the ratio of the circumference to the diameter is* 3.1416, approximately. This ratio is represented by the Greek letter π (pronounced pī), *i.e.*

$$\pi = 3.1416 = 3\tfrac{1}{7},$$

approximately. Then

$$\textit{Circumference} = 2\pi r,$$

where r is the radius of the circle.

[NOTE. The exact value of π cannot be computed. Its value to the first ten decimal places is

$$\pi = 3.1415926535.$$

Its value correct to over 700 decimal places has been computed. As a curiosity, we quote the value

$$\pi = 3.14159265358979323846264338327950288419716939937 51.]$$

The length of any arc s of h degrees is $(h/360) \cdot 2\pi r$, for we have, in Fig. 28,

$$\frac{\text{Arc } s}{\text{Circumference}} = \frac{h^\circ}{360^\circ};$$

hence

$$\textit{arc } s = \frac{h}{360} \cdot 2\pi r.$$

EXAMPLE 1. Find the circumference of a circle whose radius is 5′. By the formula, the circumference $= 2\pi \cdot 5' = 31.4'$.

EXAMPLE 2. Find the length of the arc whose central angle is 30° in a circle whose radius is 5 ft.

By definition, 30° of arc means $\frac{30}{360}$ of the whole circumference. Then, we have

$$\text{arc of } 30^\circ = \frac{30}{360} \times 2\pi \times 4 \text{ ft.} = \frac{5\pi \text{ ft.}}{6} = 2.62 \text{ ft.}$$

22. Area of a Circle. Area of a Sector. Area of a Segment. The area of any circle is equal to half the radius times the circumference, *i.e.*

$$\textit{area of circle} = \frac{1}{2} r \times 2\pi r = \pi r^2.$$

The area of a sector of a circle whose arc is h degrees (Fig. 28) is $h/360$ times the area of the circle, *i.e.*

$$\textit{area of sector} = \frac{h}{360}\,\pi r^2.$$

EXAMPLE. Find the area of a sector whose arc is 30° in a circle whose radius is 5 ft.

$$\text{Area of sector} = \frac{30}{360}\pi\ 25 = \frac{25\pi}{12} = 6.55 \text{ sq. ft.}$$

The area of a segment of a circle, as $ABsA$, Fig. 28, is the area of the sector $OAsB$ minus the area of the triangle OAB.

EXERCISES

1. What is the diameter of a wagon wheel whose circumference is 150 in.?

2. How many revolutions per second is a 30-inch automobile wheel making when the speedometer registers 25 mi. per hour?

3. What is the area of the cross section of a 6-inch stove pipe? A 3-inch one?

4. By using a carpenter's square find the diameter of a circular pipe having the same area of cross section as the sum of the areas of a 6-inch and an 8-inch circular pipe. See the accompanying figure. Do the same for two pipes whose diameters are $5\frac{7}{8}$ in. and $8\frac{3}{16}$ in., respectively.

[HINT. Use the theorem that similar surfaces are to each other as the squares of their like dimensions.]

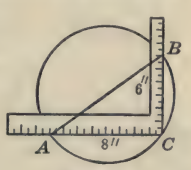


FIG. 30

5. Find the total pressure on the piston of a steam engine 8 inches in diameter, when the steam gauge registers 120 pounds per square inch.

6. A circular pond has a diameter of 75 yards. What is the cost at $3 per square yard of constructing a walk 40″ wide around it?

7. What must be the length of a binding rod for a cylindrical silo 12 feet in diameter allowing 1 ft. for overlapping?

8. Find the length of the arc and the area of the corresponding sector whose central angle is 40° in a circle of radius 10 inches.

9. Find the area of the segment of a circle whose arc is 60°, the radius being 8 inches.

[HINT. The side opposite the 30° angle in a right triangle whose acute angles are 30°, 60° is one half of the hypotenuse.]

10. *The central angle whose arc is equal to the radius* is often used as the unit of measure of angles. It is called a ***radian***. Find the number of degrees in a radian.

11. Inscribe an equilateral triangle in a circle. Show the relation between the angles of the triangle and the arcs which they subtend.

12. Has a degree of arc the same length in two unequal circles?

13. The sizes of men's hats are indicated by the diameter of a circle whose perimeter is the distance around the head where the hat rests. What size of hat does a man need if the distance around his head measures 22 inches?

14. If you wear a 7¼ hat, what is the distance around your head? Verify.

23. Polyhedrons. A ***polyhedron*** is a solid bounded by planes. A polyhedron of four faces is called a ***tetrahedron;***

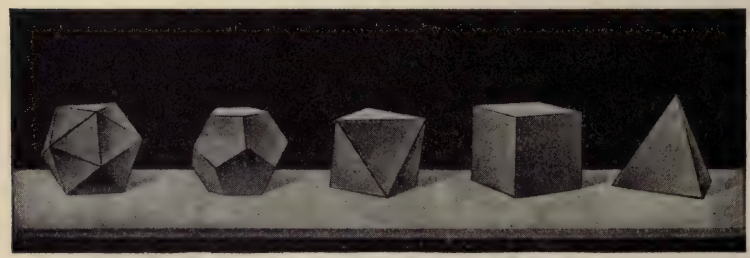


FIG. 31. — THE FIVE REGULAR SOLIDS

one of six faces, a ***hexahedron;*** one of eight faces an ***octahedron;*** etc. (Figs. 31, 32).

24. Prisms. A prism is a polyhedron, two of whose faces, called its ***bases***, are equal polygons in parallel planes, and whose other faces, called ***lateral faces***, are parallelograms

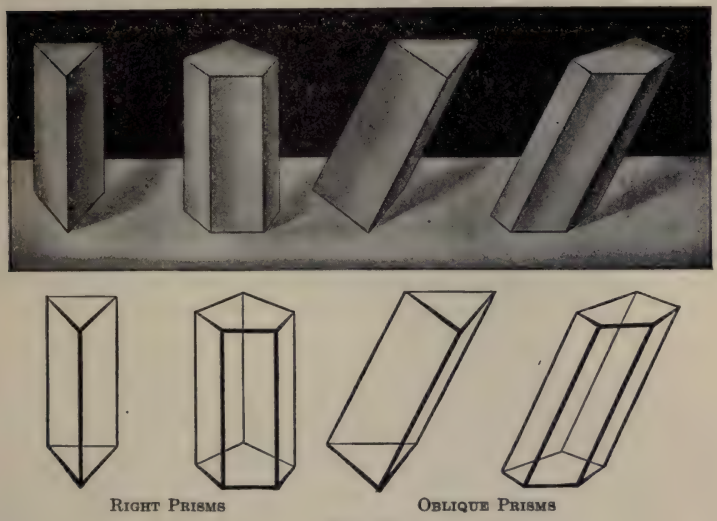


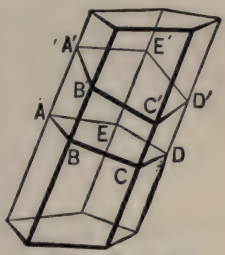
Fig. 32

whose vertices all lie in the bases. The ***altitude*** of a prism is the perpendicular distance between its bases.

A prism whose bases are perpendicular to its lateral edges is called a ***right prism*** (Fig. 32). Otherwise, the prism is ***oblique***.

Prisms are called *triangular*, *quadrangular*, etc., according as their bases are triangles, quadrilaterals, etc.

A ***right section*** of a prism is a section perpendicular to the lateral edges of the prism, as $ABCDE$, Fig. 33.

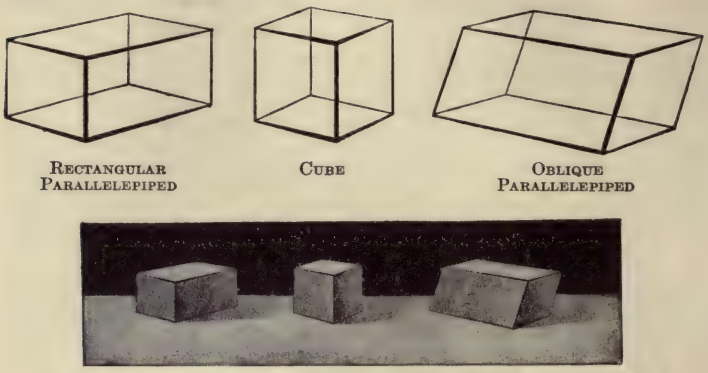


Right Section

Fig. 33

A right prism is ***regular*** if its bases are regular polygons.

25. Parallelepipeds. A ***parallelepiped*** is a prism whose bases are parallelograms.



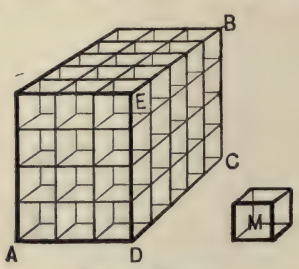
RECTANGULAR PARALLELEPIPED CUBE OBLIQUE PARALLELEPIPED

FIG. 34

A ***rectangular parallelepiped*** is one whose six faces are all rectangles. For example, most boxes are rectangular parallelepipeds.

A ***cube*** is a parallelepiped whose six faces are all squares.

26. Unit of Volume. The unit of volume is a cube whose edges are equal to the linear unit used in the measurement.



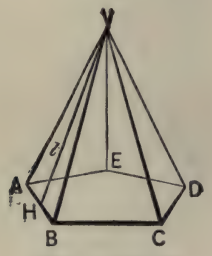
UNIT OF VOLUME

FIG. 35

The volume of any solid is the *number of units* of volume which it contains.

27. Pyramids. A ***pyramid*** is a polyhedron of which one face, called the ***base***, is a polygon and the other faces are triangles having a common vertex, called the ***vertex of the pyramid***.

The ***altitude*** of a pyramid is the length of the perpendicular let fall from the vertex to the plane of the base.



Regular Pyramid

Fig. 36

A pyramid is ***regular*** if its base is a regular polygon whose center coincides with the foot of the perpendicular let fall from the vertex to the base.

The ***slant height*** of a *regular* pyramid is the altitude of any one of the lateral faces, as VH, Fig. 36.

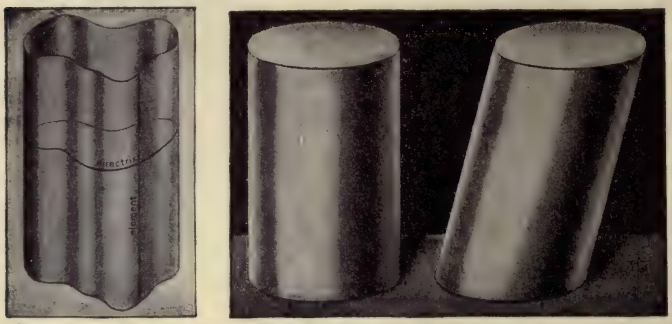
Frustum of a Pyramid

Fig. 37

A ***frustum of a pyramid*** is the portion of a pyramid included between the base and a section parallel to the base.

The ***altitude of a frustum*** is the length of the perpendicular between the planes of its bases.

28. Cylinders. A ***cylindrical surface*** is a curved surface generated by a straight line which moves parallel to a fixed straight line and constantly touches a fixed curve not in the plane of the straight line. For convenience in drawing, as



CYLINDRICAL SURFACE RIGHT CYLINDER OBLIQUE CYLINDER

FIG. 38

in the case of straight lines, it is usual to show only a limited portion of a cylindrical surface, but it really extends indefinitely in both directions.

A ***cylinder*** is a solid bounded by a cylindrical surface and two parallel plane surfaces called the ***bases***.

The ***altitude of a cylinder*** is the perpendicular distance between the planes of its bases.

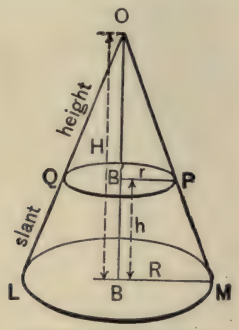
A ***right cylinder*** is one whose straight line generators are perpendicular to the bases.

29. Cones. A conical surface is the surface generated by a moving straight line which constantly touches a fixed curve and passes through a fixed point not in the plane of the curve.

A ***cone*** is a solid bounded by a conical surface and a plane which cuts all positions of the generating line. This plane surface is called the ***base of the cone***.

The ***altitude of a cone*** is the perpendicular distance from the vertex to the plane of the base.

A ***right circular cone*** is one whose base is a circle and whose altitude meets the base at the center of the circle.

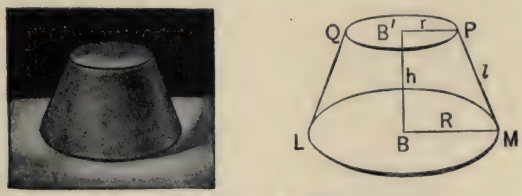


Right Circular Cone

Fig. 39

The ***slant height*** of a right circular cone is the distance along a generator from the vertex to the circle.

A ***frustum of a cone*** is the portion of a cone included between the base and a plane parallel to the base. The frustum thus has two bases. The ***altitude of the frustum*** is the perpendicular distance between these bases.



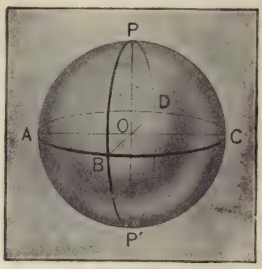
Frustum of a Cone

Fig. 40

The ***lateral surface of a frustum of a cone*** is the portion of its lateral surface included between the planes of its bases.

The ***slant height of a frustum of a right cone*** is the distance between the bases measured along a generator.

30. The sphere. A sphere is a solid bounded by a surface all points of which are equally distant from a point within



SPHERE
FIG. 41

called the ***center.*** A sphere may be generated by the revolution of a semicircle about its diameter as an axis.

31. Surface and Volume of Solids.

1. PRISM: *Volume = area of base × altitude,*
= area of right section × lateral edge.
Lateral Area of a Right Prism
= perimeter of base × altitude.

2. CUBE: *Volume* $= s^3$ where s = the length of a side.
Entire Surface $= 6s^2$.

3. CYLINDER: *Volume = area of base × altitude*
= area of right section × slant height.
Lateral Area of a Right Cylinder
= perimeter of its base × altitude.

4. PYRAMID: *Volume = one third of area of base × altitude.*
Lateral Area of a Regular Pyramid
= half perimeter of base × slant height.

5. CONE: *Volume = one third of area of base × altitude.*

Lateral Area of Right Circular Cone = half perimeter of base × slant height.

6. FRUSTUM OF A PYRAMID OR OF A CONE:

$$Volume = \frac{a}{3}(b + B + \sqrt{bB})$$

where

a = altitude of frustum,

b = area of upper base,

B = area of lower base.

Lateral Area of a Frustum of a Regular Pyramid or of a Right Circular Cone = slant height × 1/2 the sum of the perimeters of the bases = slant height × the perimeter of the mid-section.

7. THE SPHERE: $Volume = 4/3\pi r^3$

where r = the radius.

$Surface = 4\pi r^2$,

= the area of four great circles.

8. A BARREL: It may be considered as a double frustum of a cone and its volume and surface approximated accordingly. But for *circular staves* a closer approximation is the following:

$$Volume = \frac{1}{12}\pi h(2D^2 + d^2),$$

in which D = bung diameter, d = head diameter, and h = height.

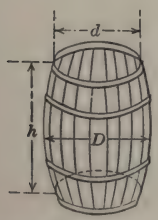


FIG. 42. — BARREL

32. Similar Figures. In general, two figures are *similar* if they have the same shape.

Two polygons are similar if they are mutually *equiangular* and if their pairs of corresponding sides are *proportional.**

Similar polyhedrons are those that have the same number of faces, respectively similar and similarly placed and have their corresponding polyhedral angles † equal.

Any two similar figures in a plane or in space can be placed in "***perspective,***" *i.e.* so that straight lines joining corresponding points of the two figures will pass through a common

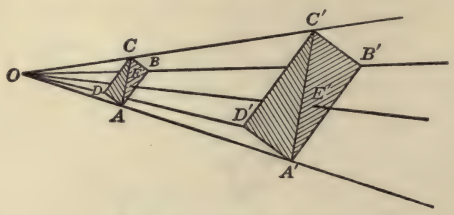


FIG. 43. — PERSPECTIVE

point, Fig. 43. Thus of two similar figures one is merely an enlargement of the other. A scale drawing, § 3, and the surface it represents are thus similar.

If *each length* in one figure is k *times* the corresponding length in the other figure, then each *area* in the first figure is k^2 *times* the corresponding area in the second. Also *each volume* in the first figure is k^3 *times* the corresponding volume in the second.

This may also be expressed as follows:

Similar surfaces are to each other as the squares of their like dimensions.

* To say that corresponding sides are proportional means that each length in one figure is a constant, κ, times the corresponding length in the other figure.

† The *opening* of three or more planes which meet at a common point is called a ***polyhedral angle.***

Similar solids are to each other as the cubes of their like dimensions.

EXAMPLE. Two oranges have diameters of 2″ and 3″. Compare their surfaces and their volumes, respectively.

Their volumes are to each other as 2^3 is to 3^3, *i.e.* as 8 to 27. Their surfaces as 2^2 to 3^2, *i.e.* as 4 to 9.

EXERCISES

1. Find the capacity in gallons of a bucket 12″ deep whose bottom and top diameters are 8″ and 10″. Find the whole surface of this bucket.

2. Find the diagonal of a barn $16' \times 30' \times 40'$.

3. What is the side of a cube whose diagonal is 2″?

4. Find the volume in cubic feet of a conical haystack whose circumference at the base is 55 ft. and whose slant height is 20 ft.

5. Find the number of gallons of vinegar in a barrel whose head diameter is 18 in., bung circumference 70 in., and height 35 in.

6. Find the volume of a regular pyramid with a square base 4 in. on each side, if the altitude of the pyramid is 12 in. Find also the lateral surface of this pyramid.

7. How much concrete is there in a circular silo whose walls are 8 in. thick, 12 ft. outside diameter and 30 ft. high?

8. How much water can be put in a round tank 8 ft. deep, if the inside diameters are 4 ft. at the bottom, and 5 ft. at the top?

9. The circumference of a sphere is 1 ft. 2 in. Find its area and its volume.

10. A farmer has two strings of drain tile 3 in. and 4 in. in diameter, respectively. He wishes to combine them into one equivalent tile. What should be its diameter? Does a tile run *full* of water?

11. What would be the size of a waterspout to be equivalent to three spouts 2 in., 4 in., and 5 in., in diameter?

12. How many 2-in. flues of an engine will equal in sectional area a smokestack 22 in. in diameter?

13. A farmer has a triangular field whose base is 20 rods and whose sides are 16 rods and 12 rods. Where, along the side 16 rods long,

should he run a fence parallel to the base in order to divide the field into two equal parts? If the fence is run parallel to the base and starts at the *middle* of the side 16 rods long, how does it divide the field?

14. In a trapezoid the parallel sides 8 ft. and 12 ft. long are 16 ft. apart. Find the altitude of the triangle formed by extending the non-parallel sides of this trapezoid. What is the area of the trapezoid?

15. In the previous exercise, where should a line be run parallel to the parallel sides to divide the trapezoidal field into two equal parts?

Ans. 7.2 rd. from the lower base.

16. Grindstones of Huron sandstone will safely stand a surface speed of 3600 ft. per minute. How many revolutions per minute (R. P. M.) may a $3\frac{1}{2}$-ft. stone be run? A 2-ft. stone? A 6-ft. stone?

17. A hollow, spherical steel shell is 1 in. thick and its outside diameter is 10 in. The specific gravity of steel is 7.8, and a cubic foot of water weighs 1000 oz. Find the weight of the shell.

18. The side of an equilateral triangle is 12 ft. Find its altitude and the area.

If the area of an equilateral triangle is 400 sq. ft., find the side.

Make a formula for area in terms of the side, x, say.

19. How much tin is there in a funnel whose diameters are $1\frac{1}{2}$ in. and 8 in. and whose height is 12 in.? What is the volume of this funnel?

20. Find the volume of a regular hexagonal pyramid 20 ft. high if each side of the hexagon is 4 ft. long. Find also the lateral surface of this pyramid.

21. How many cubic yards must be excavated in digging a ditch 200 rods long, 12 in. wide at the bottom, 6 ft. wide at the top, and a yard and a half deep?

How much water would be discharged by such a ditch in 2 hr. time if it flows half-full at the rate of 1 ft. per second?

22. A chimney is to be constructed in the form of a frustum of a circular pyramid. The height is to be 160 ft., with outside diameters 7 ft. and 10 ft. The flue is to be 6 ft. in diameter throughout the entire height. How many $2 \times 4 \times 8$ in. bricks will its construction require, allowing 10% for mortar?

23. Three oil cans are of the same height. Two of them are each 9 in. in diameter, and the other one is 13 in. in diameter. Compare the capacity of the large one with that of the other two.

24. A new grindstone is 30 in. in diameter and has a 3-in. face. What has it lost in weight when it is worn down to 26 in. in diameter, if sandstone weighs 2.42 times as much as an equal volume of water, and water weighs 62.5 lb. per cubic foot?

25. The frustum of a pyramid is 8 feet high, and its bases are equilateral triangles whose sides are each 3 feet and 4 feet, respectively. Find its volume.

26. In the previous exercise, find the volume of the entire pyramid.

27. The height of a certain frustum of a right cone is $\frac{2}{3}$ the height of the entire cone. Compare the volumes of the frustum and the cone.

28. A lateral edge of a pyramid is 6 feet. At what distance from the vertex should this edge be cut by a plane parallel to the base to divide the pyramid into two parts which are to each other as 1 to 1? As 1 to 2? As 3 to 4?

29. Find the volume of a regular tetrahedron whose edge measures 3 inches.

[HINT. The tetrahedron is a triangular pyramid.]

30. A wheat bin measures 4 ft. by 6 ft. by 10 ft. What are the dimensions of a similar bin that holds twice as much?

31. Find the area of the solid generated by an equilateral triangle that revolves about one of its sides if the length of the side is 6 inches.

32. If 0.203 of a gram of gold (specific gravity, 19.32) is plated over a spherical dome whose height is 3 cm., and the radius of whose base is 6 cm., what is the thickness of the gold?

[HINT. 1 cc. of gold weighs 19.32 grams. Therefore $.203 \div 19.32$ is the volume of the gold. This divided by the superficial area is the thickness.]

33. A steel sphere whose radius is .95 cm. weighs 28.25 grams. What is its density? *Ans.* 7.866

34. Funnels are best made with an angle of exactly 60°. If such a funnel measures 8 cm. across the top, what size filter paper will fit it flush with the edge?

35. A rectangular block of wood 7.5 by 7.46 by 3.8 cm. weighs 152.7 grams. What is its specific gravity if 1 cc. of water weighs 1 gram?

If this block is floated in water, to what depth will it sink?

33. Silos. A ***silo*** is a receptacle for preserving green feed, and ***silage*** is the material preserved. A silo is usually a cylindrical tank much higher than wide. Its size depends upon the number of animals, usually cattle, to be fed. The diameter of the silo must be of such size as to insure that the proper depth will be removed daily. Silage of all kinds deteriorates unless it be fed *regularly, evenly,* and *at a rate of not less than two inches depth daily.* Removing five or six inches daily insures but little waste of feed. Experience has shown that the most satisfactory results are obtained by providing *a horizontal feeding surface of about* 5 *square feet for each cow.* See Ex. 9 below.

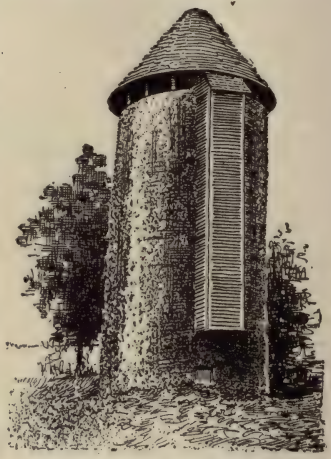


FIG. 44. — A CEMENT SILO

To insure proper settling of the silage, excluding in this way the bacteria that cause decay, *the height of the silo should seldom be less than* 30 *feet.*

On an average *one ton of silage occupies* 50 *cubic feet.*

Thus, *the diameter of a silo is controlled largely by the number of cattle to be fed, while the height is gauged by the quantity to be fed.*

One cubic foot of silage per head is a widely used daily ration.

EXERCISES

1. What should be the height of a round silo for a herd of 25 cows, if each one is to be fed 40 lb. daily for 180 days?

SOLUTION: For 25 cows a horizontal feeding surface of 5 times 25 = 125 square feet is required. This is the area of a circle 12.6 feet

in diameter. But to feed 25 cows, each 40 lb. daily for 180 days, requires

$$25 \times 40 \times 180 \text{ lb.} = 18{,}000 \text{ lb.} = 90 \text{ tons.}$$

Since 1 ton occupies 50 cu. ft., 90 tons will require 4500 cu. ft. of space. Therefore the height is the capacity, 4500 cu. ft., divided by the cross-sectional area, 125 sq. ft., *i.e.* 36 ft.

2. What should be the size of a round silo to feed 30 cows 40 lb. each daily for 150 days, 39 lb. each daily for 100 days, and 15 lb. each daily for 75 days?

3. What should be the size of a round silo for a herd of 20 cows, the ration being 40 lb. each daily for 100 days and 30 lb. each daily for 90 days?

4. Fattening steers are fed about 25 lb. each daily and calves 15 lb. each daily. How large a silo should a farmer construct to accommodate in this way 40 steers for 150 days and 20 calves for 100 days, estimating 40 lb. as the daily ration for an animal that requires 5 sq. ft. of horizontal feeding surface.

5. What is the capacity of a round silo 30 feet high and 15 feet in diameter? How many cows will it maintain? How many pounds daily can each be fed from this silo for 180 days?

6. What depth is fed daily in feeding 25 cows each 40 pounds per day? If a smaller daily ration is contemplated, how should the above rule for size of a silo be changed, in order to insure proper feeding depth daily?

7. For a herd of 15 cows, each one fed 30 pounds daily for 175 days, what should be the minimum size of the silo?

8. A silo is 16 feet in diameter and 36 feet high. What is the least number of cows that must be kept to prevent the silage from spoiling, if each cow is fed 40 pounds daily?

9. Explain the provision of 5 sq. ft. of horizontal feeding surface for each cow on the basis of a *moderate* daily feed of $33\frac{1}{3}$ lb., an *average* of 50 cu. ft. of silage per ton, and a minimum of 2″ depth removed daily.

CHAPTER III

INDIRECT MEASUREMENT — TRIGONOMETRY — SURVEYING

34. Proportion and Indirect Measurement. A ***proportion*** is an equality of ratios,* as

$$\frac{2}{3}=\frac{8}{12},\ \frac{\$3}{\$4}=\frac{12\text{ lb.}}{16\text{ lb.}},\ \frac{4\text{ hr.}}{7\text{ hr.}}=\frac{20\text{ mi.}}{35\text{ mi.}}$$

The proportion $\frac{a}{b}=\frac{c}{d}$ may also be written in the form

$$a:b=c:d.$$

There are four terms in a proportion. If any three of them are known the fourth may be found. Thus, if we have

$$\frac{x}{a}=\frac{b}{c},$$

multiplying each side by a,

$$x=\frac{ab}{c}.$$

In stating a proportion where one of the terms is unknown it is convenient to *put the unknown term first* as just shown.

To estimate the height of a barn, a farmer has an assistant

* A *ratio* is the relation of one number to another of the same kind expressed by their quotient. The ratio of a to b is a/b. Every fraction expresses the ratio of its numerator to its denominator and every integer expresses the ratio of itself to unity.

hold a staff NB vertically, Fig. 45. As he sights along AE to the top of the barn, the assistant marks the point B on

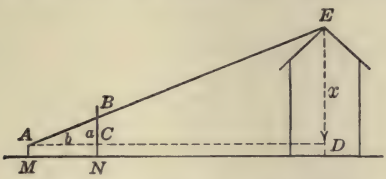


FIG. 45. — INDIRECT MEASUREMENT

the staff. When $a = BC$, $b = AC$, and AD are measured, the distance $x = DE$ may be computed from

$$\frac{x}{AD} = \frac{a}{b}. \quad \text{Why?}$$

The height of the barn is then x plus the height of the observer.

The principle used here forms the basis of nearly all practical methods of ***indirect measurement.***

EXERCISES

1. Find the height of a barn, when, as in Fig. 45, $a = 2'$, $b = 5'$, $AD = 75'$, and the observer is $5'$ tall.

2. A post $12'$ high casts a shadow $30'$ long. How high is a tree whose shadow at the same time is $160'$?

State the proposition from elementary geometry which is used in the solution of this problem.

3. At a distance of $62'$ from a building is a vertical post $10'$ high. By standing back of the post $6'$ and sighting over the end of a $4'$ vertical stick, the top of the post and the top of the building are in line. How high is the building?

4. A carpenter's square is $16'' \times 24''$, the blade is $1\frac{1}{2}''$ wide, and the arm is $2''$ wide. A man places the blade vertically on top of a post $5'$ high and $40'$ from a building. How high is the building, if the line of sight over the arm and blade just intersects the top of the building?

5. The line of sight between two objects A and B, Fig. 46, is obstructed by a building. The distance AB may now be estimated by measuring the distances AO and BO from some convenient point O, laying off $OC=OA$ and $OD=OB$, and finding DC. Show why.

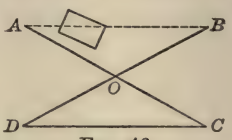


Fig. 46

6. The distance AB, Fig. 47, across a swamp or river may be estimated by use of a vertical staff to which is attached an arm CD at right angles to the staff and movable up and down along it. By sighting along EB and knowing AE, DE, DC, the distance AB is found.

If $AE=7'$, $DC=3'$, and $ED=4''$, find the distance across the swamp.

Moreover, an accessible point B' may be located so that $AB'=AB$ by simply revolving the arm DC around the staff and sighting to B' in the line EB'.

This device was used in the early days, and formed the basis of the surveying instruments used in those times.

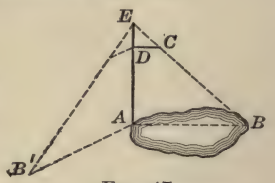


Fig. 47

7. A boy holds a pencil $6''$ long $2'$ from his eye so that it covers a chimney $200'$ distant. What is the height of the chimney? Draw a figure, and explain the process.

8. Draw a line segment. Divide it into three equal parts. Into five equal parts.

9. The following figure represents a diagonal scale used by draftsmen. The distance from 0 to 10 is $1''$.

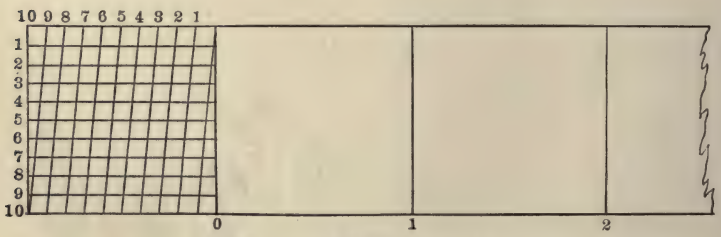


Fig. 48. — Diagonal Scale

Explain the principle involved in the use of such a scale.

Show how to set the dividers to the following distances: $.27''$, $.73''$, $1.89''$, $1.04''$.

35. Angles and the Sides of a Right Triangle. A right triangle is formed by dropping a perpendicular line from any point B in the terminal of an angle, upon the initial line at C, Fig. 49. Thus $a = BC$ is the side *opposite* the angle A,

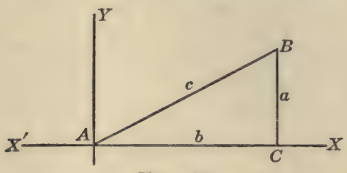


FIG. 49

$b = AC$ is the side *adjacent* to the angle A, and AB is the hypotenuse of the right triangle.

Now *as the angle A changes, the sides of this right triangle will change. The amount of turn, or the size of the angle, thus depends upon, or is related to, the sides of the right triangle,* as shown in the next few paragraphs.

36. Use of Instruments. Ratios. Angles. In the study of indirect measurement, § 34, we saw that the *ratios of sides and hypotenuse of right triangles* play an important part in solving problems involving triangles. This is particularly true in finding heights and distances and in surveying.

Where more exact methods are desired, a surveyor's transit, or a sextant, or some similar instrument, is used to find the ratio of the sides, instead of using a staff and sighting as was done in § 34. For example, suppose we wish to know the height of a tree CB, as in Fig. 50. The angle A could be measured directly by means of one of these instruments. The ratio desired, a/b in this case, is called the ***tangent;*** it

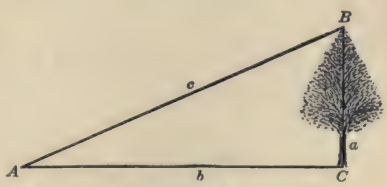


FIG. 50

can be found from tables which are specially prepared for this purpose. Suppose the angle at A is found to be 25° 20′. Then from such a table (in the column for *tangents*, Table IX, Appendix), we find opposite 25° 20′ the value .4734. This means that $a/b = .4734$. If b is measured and is found to be 100′, then $a = (.4734)100' = 47.34'$, the height required.

Moreover, *if we know the sides and therefore their ratio, such a table gives us the angle which corresponds to that ratio.* For example, if the ratio $a/b = .5467$, we find from the table of tangents that this is the ratio for 28° 40′; hence $A = 28° 40'$.

37. Trigonometric Ratios. For convenience these *ratios* in the right triangle have been given special names.

Draw a triangle ABC right-angled at C, Fig. 51, naming, as is usual, the angles with the capitals and the sides opposite the angles by the corresponding small letters.

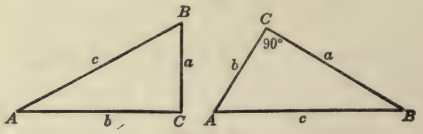


FIG. 51

DEFINITIONS.

***Tangent A** is the ratio of the side **opposite** angle A to the side **adjacent** to A, or **a/b** in Fig. 51.*

***Sine A** is the ratio of the **opposite** side to the **hypotenuse**, or **a/c** in Fig. 51.*

***Cosine A** is the ratio of the **adjacent** side to the **hypotenuse**, or **b/c** in Fig. 51.*

Abbreviations for these expressions are:

$$\tan A = \frac{a}{b}, \quad \sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}.$$

These are called ***trigonometric ratios.***

In this same triangle write the similar ratios for tan B, sin B, cos B.

EXERCISES

Find the value of the unknown term in each of the following exercises, expressing this value decimally also.

1. $5 = \frac{2}{x}$.

2. $9 = \frac{4}{a}$.

3. $27 = \frac{a}{\frac{1}{2}}$.

4. $23 = \frac{.7}{c}$.

5. $.4759 = \frac{35.6}{b}$.

6. If $\tan 32° = \frac{a}{77.3}$, find a. *Ans.* $a = (77.3) \tan 32°$.

7. Solve, as in Ex. 6, (*a*) $\sin A = \frac{18}{c}$, for c.

(*b*) $\tan 25° = \frac{210}{b}$, for b.

(*c*) $\frac{\cos 20°}{a} = 13$, for a.

8. For a right triangle ABC, with sides a, b, c, as in Fig. 51, fill in the omitted entries in the following table from the geometric properties of these special triangles. Express the ratios decimally.

Ex.	a	b	c	$\sin A$	$\cos A$	$\tan A$	angle A
(*a*)	3		6				30°
(*b*)	27	27					45°
(*c*)	15				.5		60°
(*d*)			17				30°

9. If $a = 7$, $b = 24$, find $\sin A$, $\cos A$, $\tan A$. Also find $\sin B$, $\cos B$, $\tan B$.

10. A rafter rises 5′ in a horizontal run of 12′. What is the length of the rafter? What are the sine and the cosine and the tangent of the angle which this rafter makes with the horizontal? See Fig. 11.

11. The side of an equilateral triangle is 10.4′. Find its altitude and its area. If the side is s, find its altitude and its area in terms of s.

12. What is the length of the side of an equilateral triangle which contains 1 acre?

13. Is the sine a proper or an improper fraction? Why? Is the cosine greater than unity or less than unity? How large may the tangent become?

38. Tables. The values of the trigonometric ratios for angles differing by 10′ have been computed to four decimal places and recorded in Table IX in the Appendix of this book. They are arranged with the angles from 0° to 45° *reading down the left side of the page* and the *name* of the desired functions of these angles at the *top of the page*. Angles from 45° to 90° are *read up the right side of the page* and the *name* of the functions at the *bottom* of the page.

By means of computed *corrections* these tables may be read to the nearest minute. These corrections are made according to the rule that *relatively small changes in the angle are proportional to the corresponding changes in the value.*

The use of the tables is shown by the following examples.*

EXAMPLE 1. Find sin 37° 20′. On the proper page of the table find 37° 20′ in the left-hand column marked *Degrees* at the top. Directly opposite 37° 20′ in the column marked *Sine-Value* at the top, we find .6065 Thus sin 37° 20′ = .6065 Similarly, sin 37° 40′ = .6111

EXAMPLE 2. Find sin 37° 24′. Find, in this case,

sin 37° 20′ = .6065
sin 37° 30′ = .6088

Thus 10′ change in the angle = a change of 23 in the value of the sine and therefore 4′ change in the angle = a change of $\frac{4}{10}$ of 23 = 9 in the sine, *i.e.*

	sin 37° 20′ = .6065
	correction for 4′ = 9
Hence	sin 37° 24′ = .6074

EXAMPLE 3. Find tan 37° 47′. Here

	tan 37° 40′ = .7720
	tan 37° 50′ = .7766
Therefore, for	10′, the correction is 46.
and for	7′, the correction is $\frac{7}{10}$ of 46 = 32.
Thus	tan 37° 40′ = .7720
	correction for 7′ = 32
Therefore	tan 37° 47′ = .7752

* For more extensive tables, giving *values for every minute of arc to five decimal places*, see, *e.g.*, THE MACMILLAN TABLES.

EXAMPLE 4. Find cos 19° 12′. We have

$$\begin{array}{r} \cos 19^\circ\,10' = .9446 \\ \cos 19^\circ\,20' = .9436 \\ \hline \text{correction for } 10' = 10 \end{array}$$

and correction for $2' = 2$.

This correction is *subtracted* in the case of the cosine, since the *value of the cosine becomes smaller as the angle increases.*

Thus $\cos 19^\circ\,12' = .9444$

EXAMPLE 5. If $\sin A = .6870$, find A. Not finding *precisely* this number in the table, take the one next *nearest below it.* Thus

$$\begin{array}{r} .6862 = \sin 43^\circ\,20' \\ .6870 = \sin A \\ \hline 8 = \text{correction to be found for the angle.} \end{array}$$

But $.6884 = \sin 43^\circ\,30'$.

Therefore $22 = $ correction for $10'$

and $8 = $ correction for $\frac{8}{22}$ of $10' = 4'$.

Hence $.6870 = \sin 43^\circ\,20' + 4' = \sin 43^\circ\,24'$.

EXAMPLE 6. If $\tan A = 1.6795$, find A.

$$\begin{array}{r} 1.6753 = \tan 59^\circ\,10', \\ 1.6795 = \tan A, \\ 1.6864 = \tan 59^\circ\,20'. \end{array}$$

Therefore $111 = $ correction for $10'$

and $42 = $ correction for $\frac{42}{111}$ of $10' = 4'$,

whence $1.6795 = \tan 59^\circ\,24'$, *i.e.* $A = 59^\circ\,24'$.

EXERCISES

With the table just explained, read and tabulate the sine, cosine, and tangent of each angle in the following table:

Ex.	Angle	sin	cos	tan	Ex.	Angle	sin	cos	tan
1	32° 34′				7	71° 9′			
2	27° 53′				8	27° 7′			
3	41° 22′				9	81° 29′			
4	11° 19′				10	68° 41′			
5	56° 48′				11	84° 16′			
6	61° 17′				12	64° 54′			

39. Construction of Angles. An angle may be laid off directly by means of a protractor. However, it can be constructed more accurately by means of the value of its tangent obtained from a table. When ***squared paper*** * is used neither protractor nor compasses are necessary.

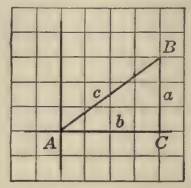


FIG. 52

EXAMPLE 1. Given, $\tan A = .75$, to construct A.

Since $.75 = \frac{75}{100} = \frac{3}{4}$, lay off 4 squares horizontally, Fig. 52, and 3 squares vertically and draw the hypotenuse AB. Then angle CAB is the required angle.

EXAMPLE 2. Construct K if $\sin K = 5/13$.

Here the hypotenuse is 13 and the side opposite K is 5. The third side of the right triangle is therefore $\sqrt{13^2 - 5^2} = 12$ and $\tan K = 5/12$. The construction then follows as just shown.

EXERCISES

Draw a right triangle to which each of the following functions belongs and give the numerical value of each of the other two functions.

1. $\tan A = \frac{7}{24}$.
2. $\cos K = \frac{9}{15}$.
3. $\sin B = \frac{1}{2}$.
4. $\tan H = 2\frac{1}{3}$.
5. $\sin F = .3$
6. $\cos A = \frac{4}{7}$.

Construct the following angles from the value of their tangents:

7. 37°.
8. 20°.
9. 54° 30′.
10. 5°.
11. 84°.
12. 60°.

40. Solution of Right Triangles. *To solve a right triangle* is to compute the unknown parts. Two fundamentally different cases exist.

I. *When two sides are given.*

II. *When a side and an acute angle are given.*

In the solution of problems, here as elsewhere, the given data should be drawn carefully to a convenient scale. This in itself is an approximate solution.

* See § 78. This is merely paper ruled into small squares or small rectangles. It can be purchased at most students' supply stores.

EXERCISES

Solve each of the following right triangles. It is assumed that the triangle is lettered as in Fig. 51 unless special lettering is given.

1. Given $c=12.5$, $A=17°\ 24'$. Find a, b, B.

SOLUTION: $B=90°-17°\ 24'=72°\ 36'$,

$$\frac{a}{12.5}=\sin 17°\ 24'=.2990;$$

hence $a=(12.5)(.2990)=3.74$

$$\frac{b}{12.5}=\cos 17°\ 24'=.9542,$$

whence $b=(12.5)(.9542)=11.9$

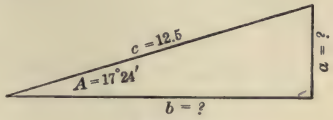


FIG. 53

2. Given $b=140$, $B=24°\ 12'$. Find a, c, A.

SOLUTION: $A=90°-24°\ 12'=65°\ 48'$.

$$\frac{a}{140}=\tan 65°\ 48'=2.2251$$

$$a=(140)(2.2251)=311.5$$

$$\frac{140}{c}=\sin 24°\ 12'=.4099,$$

whence $c=\dfrac{140}{.4099}=341.5$

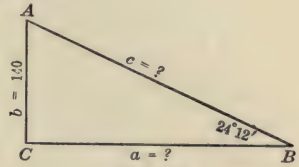


FIG. 54

3. Given $c=340$, $B=29°$. *Ans.* $A=61°$, $b=165$, $a=298$.

4. Given $b=270$, $A=54°$. *Ans.* $B=36°$, $a=372$, $c=459$.

5. Given $b=42.4$, $B=57°\ 46'$. *Ans.* $A=32°\ 14'$, $a=26.7$, $c=50.1$

6. Given $a=3$, $b=4$.

SOLUTION: $\tan A=\frac{3}{4}=.75$

$A=36°\ 52'$.

$$\frac{3}{c}=\sin 36°\ 52'=.6000,$$

whence $c=\dfrac{3}{.6}=5.$

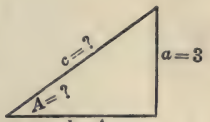


FIG. 55

Check: $a^2+b^2=c^2$, *i.e.* $3^2+4^2=5^2$.

7. Given $a=37$, $b=121$. *Ans.* $A=17°$, $B=73°$, $c=127$.

8. Given $c=5.4$, $a=2.6$. *Ans.* $A=28°\ 47'$, $B=61°\ 13'$, $b=4.7$

[HINT. Use $\sin A=a/c$.]

9. Given $c=63$, $b=47$. *Ans.* $A=41°\ 45'$, $B=48°\ 15'$, $a=42$.

10. Two sides of a triangle are 37.4 and 29.8, and their included angle is 31° 47′, as shown in the figure. Find the third side, a, and the angles B and C of this triangle and the altitude, h, on AC.

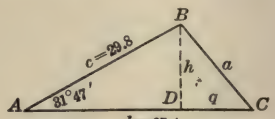


Fig. 56

[Hint. Find $p = AD$ and h in the right triangle ADB from $p = (29.8) \cos 31° 47'$, etc. Then $q = 37.4 - p$.]

11. If $c = 13.37$, $A = 13° 13'$, find a, b, B.

12. Given $a = 12.78$, $c = 13.42$ in a right triangle. Find the angles and the third side.

13. A tile drain has a fall of 4″ to the rod. What angle does it make with the horizontal in the same vertical plane with the drain?

14. What is the length of the rafters for a house 28′ 4″ wide if they have a rise of 1 in 2 of run, and project 16″ beyond the eaves? What angle does each make with the horizontal in the same vertical plane with the rafter?

15. Each of two sides of an isosceles triangle is 27 rods long and the angle at the vertex is 47° 38′. Find the base, altitude, and area.

[Hint. Bisect the given angle and solve one right triangle.]

16. A corner post is anchored to a stone buried in the ground by a stay wire which makes an angle of 25° with the horizontal. If the tension in the wire is 200 pounds, find the vertical and the horizontal pull on the stone.

17. Two sides of a parallelogram are 12′ and 15′, and the included angle is 30° 35′. Find the two diagonals.

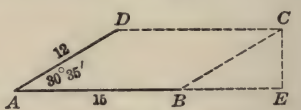


Fig. 57

[Hint. Find the horizontal and the vertical projections of BC. Then AC follows from the right triangle AEC.]

18. Two forces of 40 pounds and 50 pounds, respectively, act on a body at an angle of 40° 52′. Find the resultant of these forces and the angle it makes with either force.

[Hint. Find, as in the previous exercise, the diagonal of the parallelogram whose sides represent to scale these forces.]

19. The draft of a certain plow is rated at 296 lb. What power must be supplied to operate it if the traces of the horses pulling it make an angle of 11° with the horizontal?

41. Angles of Elevation, of Depression. The angle which a line from the eye to an object makes with a horizontal line in the same vertical plane is called an ***angle of elevation*** if the object is above the eye of the observer and an ***angle of depression*** if the object is below the eye of the observer, Fig. 58.

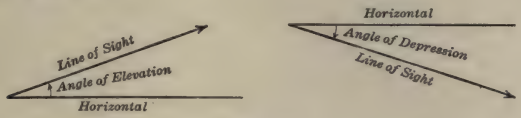


FIG. 58

EXAMPLE. The angle of elevation of the top of a barn is 42° 40′ at a point 102.4′ measured horizontally from the base. Find the height of the barn.

From the right triangle, we have

$$\frac{h}{102.4'} = \tan 42° 40' = .9217$$

Hence $h = (102.4)(.9217)$
$= 94.38'$, the height required.

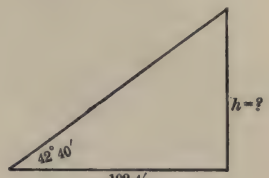


FIG. 59

EXERCISES

1. What is the angle of elevation of the sun when a 10′ pole casts a shadow 8′ long?

2. What is the height of a tree if the angle of elevation of its top is 37° 35′ measured at a point 147.8′ horizontally from its foot?

3. From the top of a cliff 167′ high the angle of depression of a man on the plain is 27° 28′. How far is the man from the base of the cliff?

4. A triangular field ABC contains one acre. If $\angle CAB = 52° 8'$ and $\angle B = 90°$, find AB and BC.

[HINT. Let $AB = x$ and find BC in terms of x from $\tan CAB$. The area in terms of x may now be found, etc.]

5. What is the rise in one third of a mile in a road which has a uniform grade of 2° 30′?

6. A tower and a building stand on a level plane. At a window 23′ from the ground, the angle of depression of the base of the tower is 13° 30′ and the angle of elevation of the top is 39° 47′. Find the height of the tower.

7. Two men are lifting a stone by means of ropes. The ropes are in the same vertical plane. What is the weight of the stone if one man pulls 105 lb. in a direction 48° with the horizontal and the other one 85 lb. in a direction 67° with the horizontal? *Ans.* 156 lb.

8. A hillside farm contains 52 acres, actual area. It is located on a slope of 16°. How much more is this than the horizontal surface underneath it and between the same vertical bounds?

Ans. 2 acres.

9. At a point A the angle of elevation of the top of a tree was 32°. At B, 160′ from A, the elevation of the tree was found to be 21°. What is the height of the tree if A, B, and the tree are in a straight line and in the same level plane?

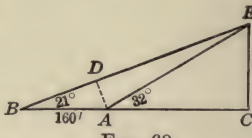


FIG. 60

[HINT. Drop a perpendicular AD on BE, Fig. 54. $\angle AED$ is equal to $32° - 21° = 11°$. Find AD from the right triangle ABD. When AD is known, compute AE from the triangle AED. Then find EC.]

10. Find the height of a precipice if its angles of elevation at two stations in a horizontal line with its base are 41° 37′ and 34° 13′ and the distance between stations is 145′.

42. Farm Surveys. Farms are surveyed by measuring distances and angles along the boundary of the farm and fixing the ***corners*** by permanent landmarks such as boulders, iron pins, concrete posts, etc. The data obtained in this way are then plotted to scale.

The area of the survey is usually computed by drawing two axes — north-south and east-west lines — so that the scale diagram of the survey lies entirely to the right and above these lines respectively. Perpendiculars are drawn from each vertex to this north-south line, *forming as many trapezoids (or triangles) as the survey has sides.*

43. Bearing of a Line or Course. Northing (Southing), Easting (Westing) of a Course. The ***bearing*** of a course means its direction by compass. N. 40° E., Fig. 61, means the direction reached after sighting north and then swinging the line of sight 40° toward the east, *i.e.* angle $NOD = 40°$.

When going around a plot or parcel of land to determine the lengths and bearings of all the sides, the surveyor bears north or south, east or west on each course, distances respectively equal to the projections of the course on these north-south, east-west lines. For example, the course OD, Fig. 61, bears north a distance OH, called the ***northing*** (***southing***), and bears east a distance HD, called its ***easting*** (***westing***). For *a closed survey*, the computation should show

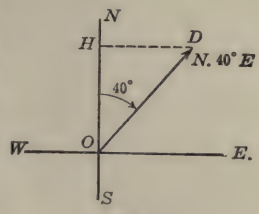


Fig. 61

$$northings = southings,$$

and

$$eastings = westings.$$

For the course OD, Fig. 61,

$$northing\ OH = OD \cos 40°,$$

and

$$easting\ HD = OD \sin 40°.$$

In general, for any course we have,

northing (southing) = length of course times cosine of its bearing,

and

easting (westing) = length of course times sine of its bearing.

Northing (or southing) of a course is the ***difference of latitudes*** of its end points, easting (westing) is the ***difference of longitudes*** of its end points.

Example 1. Plot the following courses and compute their latitudes and longitudes, *i.e.* their northings or southings, eastings or westings.

A line is run from a point A, N. 35° E., 12 chains to a point B;

from B, S. 35° E., 10 chains to a point C;

from C, S. 82° 36′ W., 12.7 chains to the place of beginning.

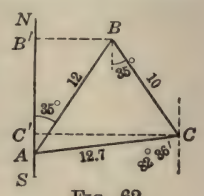


Fig. 62

Then, AB', the northing on $AB = 12 \cos 35° = 9.83$.

$B'B$, the easting on $AB = 12 \sin 35° = 6.88$

. . . . . .

CC', the westing on $CA = (12.7) \sin 82° 36' = 12.59$

This may be arranged as follows:

Course	Bearing	Distance	Latitudes		Longitudes	
			Northing	Southing	Easting	Westing
AB	N. 35° E.	12 ch.	9.83		6.88	
BC	S. 35° E.	10 ch.		8.19	5.74	
CA	S. 82° 36′ W.	12.7 ch.		1.63		12.59

Example 2. Plot the following survey to scale and find the area of the inclosed tract by projecting the courses on the north-south line through the vertex farthest west.

Beginning at a post A, thence N. 3° E., 5.60 ch. to a stone 12″ long and 5″ square, thence S. 84° E., 7.50 ch. to a post, thence S. 10° 30′ E., 6.42 ch. to an iron pin, thence N. 80° 30′ W., 9.00 ch. to the place of beginning.

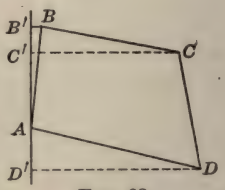


Fig. 63

The scale diagram is shown in Fig. 63. The computation may be arranged as follows:

For the first course AB, N. 3° E., 5.60 chains, we have

latitude north, $AB' = (5.60) \cos 3° = (5.60)\ (.9986) = 5.592$,

longitude east, $B'B = (5.60) \sin 3° = (5.60)\ (.0523) = .287$

These and similar results for the other courses give the following tabulation.

Course	Bearing	Distance	Computed				Balanced			
			Latitudes		Longitudes		Latitudes		Longitudes	
			Northing +	Southing −	Easting +	Westing −	N	S	E	W
AB	N 3° E	5.60 ch.	5.592		.293		5.596		.287	
BC	S 84° E	7.50 ch.		.784	7.443			.779	7.435	
CD	S 10° 30′ E	6.42 ch.		6.313	1.170			6.308	1.164	
DA	N 80° 30′ W	9.00 ch.	1.485			8.877	1.491			8.886
		28.52	7.077	7.097	8.906	8.877	7.087	7.087	8.886	8.886
				7.077	8.877					
				.02	.029					

Thus the error in latitudes is .02 ch.; in longitudes the error is .029 ch. The ***survey is balanced*** *by distributing these errors over the total perimeter*, 28.52 ch. Therefore,

$$\text{error in latitudes for 1 chain} = \frac{.02}{28.52} = .0007;$$

$$\text{error in longitudes for 1 chain} = \frac{.029}{28.52} = .001;$$

and error in AB, 5.60 ch. $= (5.60)(.0007) = .004$ for latitudes
$= (5.60)(.001) = .006$ for longitudes.

Similar results may be calculated and tabulated for the remaining courses BC, CD, DA.

Since the northings are too small, add the corrections; similarly, subtract them from the southings. Likewise, balance the eastings and westings. This gives the balanced computation in the diagram above.

The Area $ABCD = C'B'BC + CDD'C' - ADD' - ABB'$.

$C'B'BC = \frac{1}{2}\, C'B'(B'B + C'C) = \frac{1}{2}\,(.779)(.287 + 7.722) = 3.120$

$CDD'C' = \frac{1}{2}\, C'D'(C'C + D'D) = \frac{1}{2}\,(6.308)(7.722 + 8.886) = 52.382$

55.502

$ADD' = \frac{1}{2}\, AD' \cdot D'D = \frac{1}{2}\,(1.491)(8.886) = 6.625$

$ABB' = \frac{1}{2}\, AB' \cdot B'B = \frac{1}{2}\,(5.596)(.287) = .813$

7.438

Hence, **Area** $= 48.064$ sq. ch.
$= 4.8064$ A.

EXERCISES

1. Find the number of acres in a triangular field whose sides are 14.6 ch., 17.3 ch., and 19.7 ch.

2. Find the area of a triangular field that has two sides 16.6 ch. and 31.2 ch., and the included angle 63° 25′.

3. Find the area of a triangular field of which two angles are 67° 33′ and 48° 28′ and the included side is 20.4 ch.

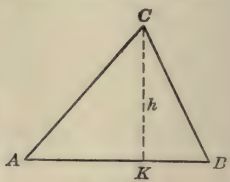


FIG. 64

[HINT. Find the third angle. Then find the altitude $h = (20.4) \sin 48° 28'$; also find AK. Since angle B is found, KB is equal to $h \div \tan B$. The area then follows from the base AB and the altitude h.]

4. A man has a triangular shaped piece of land ABC as shown in the accompanying figure. The side AB bears N. 59° 20′ E., 27.30 rd. and BC bears N. 46° 10′ W., 26.80 rd.

Find the length and the direction of AC and the area of the field assuming that the given measurements are correct.

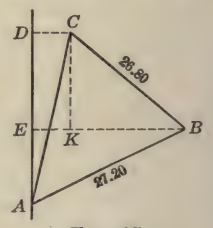


FIG. 65

[HINT. Compute the longitudes EB and BK. Their difference is CD. Similarly compute DA. But CD and DA determine the direction of AC. The area $ABC = BCDE + EBA - ACD$.]

Ans. CA bears S. 7° 8′ W.

Plot the following surveys to scale, balance the computation for distribution of error of closure, and find the area.

5. Beginning at a stone in the center of the pike; thence south 81½° east, 114 perches to a post; thence south 2° west, 62 perches to a stone; thence south 88¾° west, 113½ perches to an iron stake in the road; thence north 2° east, 80 perches to the place of beginning, containing 50 acres and 56 perches.

6. Beginning at a corner stone in the middle of the township road; thence along said road north 48° west, 13 rods and 9 links, and north 20° west, 32 rods, and north 6° east, 54 rods to a corner in said road; thence north 71½° east, 138 rods; thence south 22° east, 16 rods to corner; thence south 45° west, 171 rods to the place of beginning, containing 50 acres and 38 rods of land, be the same more or less.

7. Beginning at the southwest corner of fraction 24; thence N. $1\frac{1}{4}°$ E., 33.23 ch. to a stake in the west line of said fraction; thence S. 89° E., 8.154 ch. to a stone; thence N. $1\frac{1}{4}°$ E., 7.4 ch. to an iron pin in the north boundary line of said fraction; thence S. 89° E., 15.436 ch. to a stake; thence S. $32\frac{1}{2}°$ W., 13.50 ch.; thence S. $35\frac{1}{4}°$ E., 5.54 ch.; thence S. 48° E., 8.27 ch. to a stake; thence S. $7\frac{1}{2}°$ W., 19.20 ch. to a stake in the south boundary line of the lot; thence N. 89° W., 24.10 ch. to the place of beginning, containing 85.87 acres, be the same more or less.

44. Relations between the Sine, Cosine, Tangent. In any right triangle whose sides are a, b, c, Fig. 66,

$$a^2+b^2=c^2,$$

or, dividing by c^2,

$$\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2=1.$$

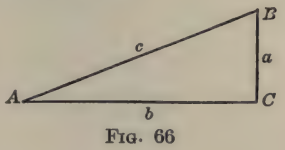


FIG. 66

But, by § 37, this is

(1) $$\sin^2 A+\cos^2 A=1.$$

Since, by definition, $\sin A=\frac{a}{c}$, $\cos A=\frac{b}{c}$, $\tan A=\frac{a}{b}$, it follows, by dividing, that

(2) $$\frac{\sin A}{\cos A}=\tan A.$$

EXERCISES

1. Construct a right triangle whose sides are 3, 4, 5. Find the value of the sine, cosine, and tangent of each acute angle.

Illustrate formulas (1) and (2), § 44, by use of this triangle.

2. Proceed as in Ex. 1 for a right triangle whose sides are 8, 15, 17. For one whose sides are 7, 24, 25.

3. Proceed as in Ex. 1 for a right triangle that has its hypotenuse $c=10$, and an acute angle $=53°\ 8'$.

4. State $\sin A$ in terms of $\cos A$. Also $\cos A$ in terms of $\sin A$.

5. Given $\sin A=.6$. Find $\cos A$ and $\tan A$.

[HINT. Construct a right triangle with one leg 6 and the hypotenuse 10. Find the other leg.]

6. Given $\tan A = 1$. Find $\sin A$ and $\cos A$.

7. Given $\tan A = 1.732$ Use a table to find $\sin A$, $\cos A$.

Show that the following relations are true by changing one or both sides of the relation, making use of (1), (2), § 44.

8. $(\sin A + \cos A)^2 = 1 + 2 \sin A \cos A$.

[HINT. Square the parenthesis and reduce by (1), § 44.]

9. $\left(\frac{1}{\cos A} + \tan A\right)\left(\frac{1}{\cos A} - \tan A\right) = 1$.

10. $\cos A \cdot \frac{1}{\sin A} \cdot \tan A = 1$.

11. $\sin^2 A(1 + \tan^2 A) = \tan^2 A$.

12. $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A$.

13. $(\sin A + \cos A)^2 + (\sin A - \cos A)^2 = 2$.

14. $\frac{1 - \sin A}{\cos A} = \frac{\cos A}{1 + \sin A}$.

[HINT. Clear the equation of fractions.]

45. Sine, Cosine, Tangent of Complementary Angles. In any right triangle, Fig. 67, $A + B = 90°$.

By definition.

$$\sin A = \frac{a}{c} = \cos B,$$

$$\sin B = \frac{b}{c} = \cos A,$$

$$\tan A = \frac{a}{b} = \frac{1}{\tan B}.$$

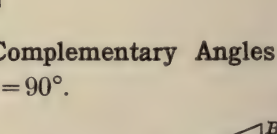


FIG. 67

Thus, *the sine of an acute angle equals the cosine of its complement*, etc.

EXERCISES

1. Find sin 50°, cos 50°, tan 50° in terms of functions of 40°.

2. Express each of the following as a function of the complementary angle.

(*a*) sin 35°. (*c*) cos 47°. (*e*) tan 60°.
(*b*) tan 20°. (*d*) cos 75°. (*f*) sin 57° 24′.

CHAPTER IV

REVIEW OF ALGEBRA

46. Use of Letters. Historically, the first use of algebra was to abbreviate explanations of arithmetic problems. Its modern use, however, is much broader. It includes the free use of symbols to state rules and to discuss problems and numbers which are stated in words.

For example, the area of a triangle is the base multiplied by one half the altitude. This may be stated briefly as

$$A = \tfrac{1}{2}\, ab,$$

where A = area, a = altitude, b = base.

47. Equations. An equation is a statement that one expression is equal to another. This statement is written with the ***sign of equality***, $=$, between the expressions.

Thus the equation $3x+4=15$ is read, $3x$ plus 4 equals, or is equal to, 15.

If a value of an unknown quantity is found which makes the two sides of a given equation equal, it is said to ***satisfy the equation.*** This value is called a ***solution*** or a ***root*** of the equation.

For example, if $x+4=5$, the value $x=1$ satisfies this equation, since, when $x=1$, both sides of the equation are equal.

To determine finally whether a given value is a solution, the value should be substituted in the equation to see if it does satisfy it.

EXAMPLE 1. Solve the equation

$$3x+4=10.$$

Subtracting 4 from each side of this equation gives

$$3x=10-4,$$

or

$$3x=6.$$

Dividing each side by 3 gives

$$x=2.$$

This is the solution of the given equation. Substituting 2 for x in the given equation, we have

$$6+4=10.$$

This means that the equation is satisfied, and hence 2 is a solution.

EXAMPLE 2. Solve the equation

$$\frac{5x}{6}-3=\frac{x}{4}-\frac{29}{12}.$$

Multiplying both sides of this equation by 12, the least common multiple of the denominators, gives

$$10x-36=3x-29.$$

Subtracting $3x$ from each side and then adding 36 to each side, we find

$$10x-3x=36-29,$$

or

$$7x=7,$$

whence

$$x=1.$$

Substituting $x=1$ in the given equation gives

$$\tfrac{5}{6}-3=\tfrac{1}{4}-\tfrac{29}{12},$$

or

$$-\tfrac{13}{6}=-\tfrac{26}{12}=-\tfrac{13}{6}.$$

Therefore $x=1$ is a root of the equation.

48. Operations on Equations. Without disturbing the equality we may

(1) *Add or subtract the same number* on each side of the equation.* This is usually performed by actually moving it to the other side of the equation and changing its sign.

(2) *Multiply or divide both sides of the equation by the same number,* except that division by zero is excluded.*

* If any expression other than a simple number is used, it must at least represent a number.

EXERCISES

Solve each of the following equations to find the value of the unknown letter or term. Check each solution.

1. $2x+3=7$. **4.** $-2x+3=-7$. **7.** $12\sin A+1=7$.

2. $2x-3=7$. **5.** $7x-15=3x-7$. **8.** $2\tan A+3=1+4\tan A$.

3. $2x+3=-7$. **6.** $6-5a=3-a$. **9.** $\frac{7}{12}x-\frac{1}{4}=2x-\frac{5}{3}$.

10. The number of cows and sheep in a certain farmyard is 45, and there are four times as many sheep as cows. Find the number of each.

11. One angle is twice the size of another angle. If their sum is the complement of three times the larger one, find each of the two angles. Solve the similar problem obtained by substituting the word *supplement* for the word *complement*.

12. A does $\frac{3}{5}$ of a piece of work in $10\frac{1}{2}$ days. With the help of B they then finish the work in 4 days. How long would it take B alone to do the work?

13. Find the angle between the hands of a clock at quarter past four o'clock. At what time will they be exactly together?

14. The driver of a milk wagon is charged \$28.56 for the entire load. He carries certified milk at 22¢ per quart and 5 times as much milk in cans at 16¢ per quart. How much of each does he carry?

15. Find four consecutive integers whose sum is 94.

16. What are the dimensions of a right triangle whose area is 1.2 acres if the height is 3.7 times the base?

17. In a triangle one exterior angle is 108°. If the ratio of the other two angles is 2.6, find the three angles of the triangle.

18. A miller has wheat worth \$2.20 per bushel and another lot worth \$3.00 per bushel. He wishes to mix these to make 20 bushels of wheat which shall be worth \$2.40 per bushel. How much of each shall he take?

19. A farmer has a cow whose milk contains $3\frac{1}{2}\%$ of butter fat (*called a* $3\frac{1}{2}\%$ *milk*) and another one which gives a 5% milk. How shall he mix them to obtain 50 lb. of a 4% milk?

20. A man can seed a wheat field in 8 days. How long would it take him to do it with the aid of a helper who can do only $\frac{2}{3}$ as much as he can?

49. Addition. In business and in ordinary life it is often necessary to add positive and negative numbers or expressions. To add $8x$ and $-5x$ is merely to subtract $5x$ from $8x$. Longer expressions are added by grouping *similar* terms.*

EXAMPLE. Add $6x-8y+5z$
and $3x+4y-6z$
the sum is $9x-4y-z$

EXERCISES

Find the sum of the following:

1. $2a+3b-4c$, $5c-4a-b$, and $b+5a-2c$.
2. $4x-5y-2z$, $-2x+3y-7z$, and $-9x-y+4z$.
3. $3A-B+2C$, $5A-2B+C$, and $-A+3B-4C$.
4. $5.6x-2.7y$, $3.5x+4.2y$, and $-2.2x-6.5y$.
5. $5x^2-2.5m$, $-.5x^2+4m$, and $1.45x^2+3.2m$.

50. Subtraction. To subtract one quantity from another, we proceed as in addition *after changing the sign of the quantity to be subtracted.*

EXAMPLE. From $4x+y-10z+2$
take $-x-6y+11z-3$
the difference is $5x+7y-21z+5$

EXERCISES

1. From $6a-7b-10c$ take $12a+5b-6c$.
2. From $15x-6-2y$ take $-5y+9-x$.
3. From $2-7A^2+A$ take $3A-5A^2+8$.
4. From $1.5E-E^2+2.8$ take $.9E-3E^2-1$.
5. From $20r-12.5s+4.2t$ take $6t-2.4r+9s$.
6. $a^2+24a-10-(16-5a+a^2)=?$
7. $x^2-5x+7-(-2x^2-5x+2)=?$
8. Simplify $16-2\sqrt{x}+5\sqrt{x}-24-7\sqrt{x}$ by combining similar terms.

* *Similar terms* are those that have a common factor. For example $2x$, $3x$, ax, mnx are all similar with respect to x.

51. Removal of Signs of Grouping. When a *negative sign* precedes a parenthesis or other grouping sign, change all positive or negative signs within, including the sign of the first term, when the sign of grouping is removed. But when a parenthesis is preceded by a *positive sign*, no change is made in the terms when the grouping sign is dropped.

EXAMPLE. $3x-(5x-2y)=3x-5x+2y.$

$$-(P+Q-R)=-P-Q+R.$$

EXERCISES

Remove the grouping signs and add similar terms in each of the following expressions.

1. $8x-(7-2x)$.
2. $(16x-7y)-(x-4y)$.
3. $(2p-5q)+(4p+q)+6Q$.
4. $A-(3B-2.5)-(B-1.5A)$.
5. $2a-(b-(2a-b))$.
6. Solve $6y-(4-2y)=20$ for y.
7. Solve $12.5x-(10-2x)=19-(x-2)$ for x.
8. Solve $7x-[x-(5x-7)]+3=24x-9(1+x)$ for x.

52. Multiplication. The product of two numbers having like signs is positive, of two having unlike signs, negative.

For example, three times 5 may be written

$$(3)(5)=5+5+5=15.$$

In the same way $(3)(-5)=(-5)+(-5)+(-5)=-15.$

The multiplier $+3$ is here a positive integer and the multiplicand is added three times. If the multiplier is negative, the multiplicand should be subtracted. Thus

$$(-3)(5)=-(+5)-(+5)-(+5)=-15,$$

and $(-3)(-5)=-(-5)-(-5)-(-5)=-(-15)=+15.$

Thus, generally,

$$(a)(b)=ab \quad \text{and} \quad (-a)(-b)=ab,$$

$$(a)(-b)=-ab \quad \text{and} \quad (-a)(b)=-ab.$$

53. Exponents. When some of the factors of a number are alike, it is often convenient to indicate the fact as follows:

$$2\times4\times4\times4=2\times4^3=128,$$
$$3\ a\ a\ y\ y\ y=3\ a^2y^3.$$

The small figure written above and to the right of an expression denotes the number of equal factors and is called an ***exponent.***

EXERCISES

Multiply

1. $4\ xy$ by $5\ xy$.
2. $2\ abc$ by bc.
3. x^8 by x^3.
4. $3\ b^4c$ by $4\ abc^3$.
5. $7\ r^7$ by $2\ s^3$ by $3\ r^2s^2$.
6. am^2cd^3 by $-a^2md^2$.
7. $-4\ a^2xy$ by $-3\ ax^2y^3$.
8. x^k by x^p.
9. a^x by a^y.
10. 3^{2b} by 3^{2c}.

Carry out each of the following indicated multiplications.

11. (*a*) $2(3+5)$. (*b*) $2(x+y)$. (*c*) $4\ a(a+b+ac)$.
12. $x(x^2+1)$.
13. $x^3(2\ x^4-3\ x^3+4\ x)$.
14. $a^2(a^2-a+1)$.
15. $x^n(x^{1-n}+x^{3+n})$.
16. Multiply x^2+xy+y^2 by $x-y$.
17. What are the factors of a^3-b^3 as shown by Ex. 16?
18. Multiply a^2-ab+b^2 by $a+b$.
19. Multiply x^2+xy+y^2 by x^2-xy+y^2.
20. Multiply $3\ ab-2\ a^2-2\ a$ by $2\ ab-2\ a-3\ b-6$.

54. Standard Type-forms. Multiplications may frequently be performed mentally. Some standard type-forms for this purpose are the following:

$$(a+b)(a-b)=a^2-b^2.$$
$$(a+b)(a+b)=a^2+2\ ab+b^2.$$
$$(a-b)(a-b)=a^2-2\ ab+b^2.$$
$$(a+b)^3=a^3+3\ a^2b+3\ ab^2+b^3.$$
$$(a-b)^3=a^3-3\ a^2b+3\ ab^2-b^3.$$
$$(a+b+c)^2=a^2+b^2+c^2+2\ ab+2\ ac+2\ bc.$$
$$(x+a)(x+b)=x^2+(a+b)x+ab.$$

EXERCISES

1. State in words the type-forms given above.

Carry out orally each of the following indicated multiplications.

2. $(2x-3y)(2x+3y)$.
3. $(x-3)(x+3)$.
4. $(x^2y-3b)(x^2y+3b)$.
5. $(a+3b)(a-3b)$.
6. $(x+[a+b])(x-[a+b])$.
7. $(x-2y)^2$.
8. $(3x+2y)^2$.
9. $(x+2y)^3$.
10. $(1-a)^3$.
11. $(a+b+2)^2$.
12. $(a-b-c)^2$.
13. $(2x-a+3)^2$.
14. $(x+7)(x+2)$.
15. $(a-3)(a+5)$.
16. $(x+2)(x-3)$.
17. $(3ab-x)(3ab+7x)$.
18. $(p^2q-10)(p^2q+3)$.
19. $(x^2-3)(x^2-4)$.

55. Division. The rules for signs hold in division as in multiplication, as shown in § 52.

EXERCISES

1. Divide $y^2+15y+36$ by $y+4$.
2. Divide $x^4+x^3+7x^2-6x+8$ by x^2+2x+8.
3. Divide A^3-1 by $A-1$.
4. Divide $a^3+b^3-c^3+3abc$ by $a+b-c$.
5. Divide $1+x$ by $1-x$. How many terms in the quotient?
6. Divide 1 by $1+x$ to four terms.

56. Factoring. The word factor means "maker." Factors are thus merely numbers or expressions which make other numbers or expressions when multiplied together. The factors of 6, for example, are 2 and 3, since 2 times 3 is equal to 6.

Factoring is the process of finding the factors of an expression. Some expressions occur quite often in the applications of algebra. The different cases may be grouped under so-called type-forms, such as *taking out a common factor*, *grouping similar terms*, the *difference of two squares*, etc.

The following type-forms will aid in finding the factors of many expressions.

$$ab+ac=a(b+c).$$

$$a^2-b^2=(a+b)(a-b).$$

$$a^2+2\,ab+b^2=(a+b)^2.$$

$$a^2-2\,ab+b^2=(a-b)^2.$$

$$a^3-b^3=(a-b)(a^2+ab+b^2).$$

$$a^3+b^3=(a+b)(a^2-ab+b^2).$$

$$x^2+(a+b)x+ab=(x+a)(x+b).$$

EXAMPLES. $A^2-9\,x^2=(A+3\,x)(A-3\,x).$

$$9\,x^2-30\,xy+25\,y^2=(3\,x-5\,y)^2.$$

EXERCISES

Factor each of the following expressions by inspection.

1. $a^2b^2-x^2$.
2. $(x+y)^2-z^2$.
3. $a^2-(b+c)^2$.
4. $a^2-(b-c)^2$.
5. $9\,x^2+6\,x+1$.
6. $x^2+4\,xy+4\,y^2$.
7. $x^2+7\,x+12$.
8. $a^2+4\,ab-21\,b^2$.
9. x^3+1.
10. x^4-y^4.
11. $8\,x^3+27$.
12. $a^3-27\,b^6$.

Factor each of the following expressions by grouping.

13. $ax-ay+bx-by=a(x-y)+b(x-y)=(a+b)(x-y)$.
14. $y^3+3\,y^2+3\,y+1$.
15. $a^5-ab^4-a^4b+b^5$.
16. In $x^2+6\,x+k$, what is the value of k to make the expression a perfect square?
17. How much must be added to $x^2+8\,x$ to make it a perfect square? How much to $x^2+5\,x$ to make it a perfect square?

Resolve each of the following expressions into factors.

18. $a^2+6\,a+8$.
19. $x^3-4\,x^2+3\,x-12$.
20. $3\,a^4-8\,a^2-35$.
21. $1-12\,ab+35\,a^2b^2$.
22. $48\,r^2-10\,r-25$.
23. $15\,p^2-pr-2\,r^2$.
24. $4\,x^2-13\,x+10$.

57. The Binomial Theorem. From the following equalities we may observe the law for *raising a binomial to positive, integral powers by inspection.*

$$(a+b)^2 = a^2+2\,ab+b^2.$$
$$(a+b)^3 = a^3+3\,a^2b+3\,ab^2+b^3.$$
$$(a+b)^4 = a^4+4\,a^3b+6\,a^2b^2+4\,ab^3+b^4.$$
$$(a+b)^5 = a^5+5\,a^4b+10\,a^3b^2+10\,a^2b^3+5\,ab^4+b^5.$$

We may now make the following observations.

(1) The number of terms on the right of the equality sign is always one more than the exponent on the left.

(2) The exponent of the first letter in the first term of the expansion is the same as the power on the left. This exponent *decreases* by 1 in each succeeding term. The exponent of the second letter, b, in the second term is 1, and *increases* by 1 in each succeeding term.

(3) The *first coefficient* is 1; the second coefficient is exactly the power to which we are raising the binomial; and in general, *the coefficient of any term may be found from the preceding coefficient by multiplying it by the exponent of a in that term and dividing this by one more than the exponent of b in that term.*

For example,

$$(a+b)^{10} = a^{10}+10\,a^9b+\frac{10\cdot 9}{1\cdot 2}\cdot a^8b^2+\cdots$$
$$= a^{10}+10\,a^9b+45\,a^8b^2+\frac{45\cdot 8}{3}a^7b^3+\text{ etc.}$$
$$(a+b)^n = a^n+na^{n-1}b+\frac{n\cdot(n-1)}{1\cdot 2}\,a^{n-2}b^2+\text{ etc.}$$

These facts make up the ***binomial theorem.*** It is true for all values of a and b and for n equal to any positive integer.*

* The binomial theorem may be extended so that, under certain restrictions, it will hold for other values of n, *e.g.* fractional and negative values of n.

EXERCISES

Expand each of the following binomials as indicated.

1. $(a+b)^7$.
2. $(a+b)^{11}$.
3. $(a+b)^{15}$.
4. $(a-2)^3$.
5. $(a-2)^4$.
6. $(x+\sqrt{a})^2$.
7. $(x-\sqrt{a})^3$.
8. $\left(\frac{x}{2}+\frac{y}{3}\right)^3$.
9. $\left(\frac{x}{2}-y^2\right)^4$.

10. Given two line-segments a and b. Construct a square on the segment $a+b$. Show by means of this square that $(a+b)^2=a^2+b^2+2\,ab$.

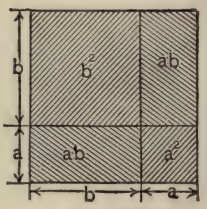


FIG. 68

58. Fractions. A ***fraction*** is an indicated division of one number or expression called the ***numerator*** by another number or expression called the ***denominator***.

As in elementary arithmetic, the *form* of a fraction may be changed without thereby changing the value of the fraction.

The value of a fraction is unchanged if both numerator and denominator are multiplied by the same expression, or divided by the same expression, excepting zero.

Fractions may be ***reduced to lowest terms*** by dividing both numerator and denominator by their common factors.

EXERCISES

Reduce to lower terms each of the following expressions.

1. $\frac{6}{32}$.
2. $\frac{185}{725}$.
3. $\frac{3^3 4^2\,a^2b^3}{2^2 3\,ab^2}$.
4. $\frac{35\,ax^2y^3}{28\,a^4x^3y}$.
5. $\frac{a^2b-ab^2}{a^2b^2}$.
6. $\frac{-39(p-q)(r-s)^3(t-1)^2}{28\,(p-q)(s-r)(1-t)}$.
7. $\frac{15\sin^2 x\tan x}{20\sin x\tan^3 x}$.
8. $\frac{x^2-y^2}{x-y}$.
9. $\frac{ax+ay-x-y}{x^2-y^2}$.
10. $\frac{x^3-8}{x^2+x-6}$.
11. $\frac{x^4-y^4}{x^2-4\,xy+3\,y^2}$.
12. $\frac{\sin^2 x-\cos^2 x}{\sin x+\cos x}$.
13. $\frac{6\,s^2-s-2}{9\,s^2-4}$.
14. $\frac{12\,a^2-ax-6\,a^2}{8\,a^2-2\,ax-3\,a^2}$.

59. Addition and Subtraction of Fractions. *In order to add or subtract fractions they must be reduced, as in elementary arithmetic, to the same denominator.* This denominator should be the *least* one common to all of the denominators.

Thus $$\frac{2}{3}+\frac{5}{7}=\frac{14}{21}+\frac{15}{21}=\frac{29}{21}.$$

Similarly, $$\frac{a}{b}+\frac{x}{y}=\frac{ay}{by}+\frac{bx}{by}=\frac{ay+bx}{by}.$$

EXERCISES

Perform the operations as indicated in each of the following expressions.

1. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}$.
2. $\frac{2}{3}+\frac{8}{15}-\left(\frac{17}{35}+\frac{7}{9}\right)$.
3. $\frac{x-2}{3}+\frac{2x+3}{2}$.
4. $\frac{1}{a}+\frac{1}{b}$.
5. $\frac{1}{a}-\frac{1}{b}$.
6. $\frac{1}{a+b}+\frac{1}{a-b}$.
7. $\frac{1}{a+b}-\frac{1}{a-b}$.
8. $\frac{x}{x-1}+\frac{x}{x+1}$.
9. $\frac{x}{x-1}-\frac{x}{x+1}$.
10. $\frac{1}{a}+\frac{2}{a^2}+\frac{3}{a+1}$.
11. $4ab+\frac{1}{(a-b)^2}$.
12. $\frac{3}{2}+\frac{5a}{a-2}+\frac{5}{(a-2)^2}$.
13. $\frac{5}{x+2}-\frac{7}{x-1}+\frac{1}{x^2+x-2}$.
14. $\frac{x-2}{x-1}-\frac{x-1}{x-2}+\frac{3}{x^2-3x+2}$.
15. $3-\frac{5(a+b)}{a-b}+\frac{(a+b)^2}{(a-b)^2}$.

60. Multiplication and Division of Fractions. *As in arithmetic, **to multiply one fraction by another** multiply their numerators together for the numerator of the product and their denominators together for the denominator of the product.* Any factors common to numerator and denominator may be cancelled before multiplying.

***To divide one fraction by another**, invert the divisor and then multiply.*

When the division of one fraction by another is indicated in the form of a fraction, this fraction is called a ***complex fraction.***

Thus, $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \times \dfrac{d}{c} = \dfrac{ad}{bc}$; $\qquad \dfrac{\frac{2}{3}}{\frac{6}{5}} = \dfrac{2}{3} \times \dfrac{5}{6} = \dfrac{5}{9}$.

EXERCISES

Perform each of the following indicated operations.

1. $\frac{2}{3} \times \frac{3}{4} \times \frac{6}{7} \times \frac{8}{9}$.
2. $\frac{15\,a^2}{27\,b^2} \times \frac{-117}{35} \times \frac{18\,b}{-39\,a}$.
3. $\frac{x-3}{2} \times \frac{4}{x^2-2\,x-3}$.
4. $\frac{x-y}{x-2\,y} \times \frac{x^2-4\,y^2}{x^3-y^3}$.
5. $\frac{v^2+6\,v+9}{16-8\,v+v^2} \times \frac{v-4}{9-v^2}$.
6. $\left(1+\frac{a}{b}\right) \times \left(\frac{b^2}{a^2-b^2}\right)$.
7. $\left(1-\frac{x^2}{g^2}\right) \times \frac{1}{x-y}$.
8. $\left(1+\frac{1}{x}+\frac{1}{x^2}\right) \times \frac{8\,x^3}{x^3-1}$.
9. $\frac{15\,pq^2}{32\,r} \div \frac{25\,qr}{36}$.
10. $\frac{4\,x^2}{-ay^2} \div \frac{-12\,ax}{9\,y}$.
11. $\frac{2\,t^2}{a^3+t^3} \div \frac{t}{a+t}$.
12. $\left(1-\frac{x^2}{y^2}\right) \div \left(\frac{x+y}{xy}\right)$.

Simplify the following expressions.

13. $\dfrac{\frac{5}{2}}{\frac{3}{3}}$.
14. $\dfrac{\frac{4}{7}}{\frac{2}{5}}$.
15. $\dfrac{1+\frac{a}{b}}{\frac{a}{b}-1}$.
16. $\dfrac{a+1}{1+\frac{1}{a}}$.
17. $\dfrac{1-\frac{2\,x}{a}+\frac{x^2}{a^2}}{1-\frac{x^2}{a^2}}$.
18. $\dfrac{\frac{x+y}{x}+\frac{x-y}{y}}{\frac{x-y}{x}-\frac{x+y}{y}}$.

61. Simple Linear Equations of One Unknown Quantity. A ***linear equation*** is one in which there is no term of higher degree than the first.

We shall now study some equations of the first degree which are more complicated than those given earlier in this

chapter but which may be solved by the same methods used there.

Equations may be ***cleared of fractions*** by multiplying both sides of the equation by the least common multiple of the denominators.

EXAMPLE. Solve the following equation for x:

$$\frac{6x-3}{2x+1} = \frac{3x-2}{x}.$$

The equation is cleared of fractions by multiplying both sides by the least common denominator, $(2x+1)(x)$. This gives

$$x(6x-3) = (3x-2)(2x+1),$$

or

$$6x^2-3x = 6x^2-x-2.$$

Collecting the like terms, we find

$$-2x = -2$$

and thus

$$x = 1.$$

The solution may be checked by substituting 1 for x in the original equation.

EXERCISES

Solve each of the following equations.

1. $4\left(2x-\frac{1}{2}\right)-6\left(\frac{x}{2}-\frac{1}{3}\right)=.2$

2. $4(x-\frac{1}{2})^2+6(x-\frac{2}{3}) = (2x-5)^2+302.$

3. One side of a right triangle is 5 inches and the other side is 2 inches shorter than the hypotenuse. How long is the hypotenuse?

4. When the radius of a circular plot of ground is increased 2 feet, the area is increased 88 sq. ft. Find the size of the original plot, using $\pi = 22/7$.

5. At what time between 3 and 4 o'clock are the hour and minute hands of a clock together?

6. Calculate when you *were, or will be,* half as old as your father.

Solve each of the following equations, the last letters of the alphabet x, y, z, w, etc. being the unknowns.

7. $\frac{2}{x}+3=4.$

8. $\frac{4}{3x+2}=\frac{3}{5}.$

9. $(a+b)(x+2)=c.$

10. $\frac{3+\frac{2x+1}{4}}{3-\frac{2x+1}{4}}=3.$

11. $ax+bx=c+d.$

12. $y=\frac{a(y+1)}{b}.$

13. $(a+z)(k-z)-(a-z)(c+z)=2.$

14. Solve the equation

$$h=\frac{k+i}{l+i}$$

for i. Also solve it for l.

15. Solve the following equation for F:

$$C=\tfrac{5}{9}(F-32).$$

This equation represents the relation between the readings on Centigrade and Fahrenheit thermometers. What is the Centigrade reading for 68° Fahrenheit? Change 25° C. to Fahrenheit reading.

16. A farmer made 100 lb. of fertilizer, using 95 parts acid phosphate, 100 parts cottonseed meal, and 5 parts muriate of potash. He then decided to add enough acid phosphate to each 100 lb. of the original mixture to make the acid phosphate content 50% of the total. How much should he add? *Ans.* 5 lb.

[Hint. Let x = the number of pounds added. Then

$$50\% \text{ of } (100+x)=47\tfrac{1}{2} \text{ lb.}+x.]$$

17. In Ex. 16, how many pounds less of acid phosphate should be used to every 15 pounds of muriate of potash, in order to reduce the acid phosphate content of the mixture to 40%?

18. In the fertilizer of Ex. 16, how many pounds less of acid phosphate shall be used to every 60 pounds of cottonseed meal in order to reduce the acid phosphate content of the mixture to 42.5%?

19. A fertilizer used in cotton fields consists of 62½% acid phosphate, 30% dried blood, and 7½% muriate of potash. It is desired to raise the acid phosphate content to 70%. How many pounds of acid phosphate must be added to one ton to do this?

62. Quadratic Equations. A *quadratic equation* is one in which there is a term of the second degree, but none of higher degree. There may be terms of lower degree.

For example,

$$3x^2-11x+2=0$$

is a quadratic as it contains terms of the second and lower degree.

Quadratics may be solved

(*a*) *by factoring*, or

(*b*) *by completing the square.*

EXAMPLE. Solve the equation $3x^2+6=11x$ by factoring.

Transpose all terms to the left side of the equation. Thus

$$3x^2-11x+6=0.$$

Factoring the left-hand side, we obtain

$$(3x-2)(x-3)=0.$$

But this means that each factor may be put equal to zero, since a product is zero if any of its factors are zero. Thus either

$$3x-2=0, \text{ } i.e. \text{ } x=\tfrac{2}{3},$$

or else

$$x-3=0, \text{ } i.e. \text{ } x=3.$$

Check: Substituting $\frac{2}{3}$ for x in the original equation gives

$$3\cdot\tfrac{4}{9}+6=11\cdot\tfrac{2}{3}, \text{ or } \tfrac{12}{9}+6=\tfrac{22}{3}.$$

Substituting 3 for x in the same equation gives

$$3\cdot 3^2+6=33.$$

EXERCISES

Solve each of the following quadratics by factoring.

1. $5x^2+6x=32$.

2. $3x^2-44x=-121$.

3. $4x^2+5x=0$.

4. $x^2-2bx=2ax-4ab$.

5. $3x^2=12$.

Notice, in this example, *the double sign obtained on extracting the square root.*

6. $11x^2-10=5x^2+44$.

7. $(2x+3)^2=5x+39$.

63. Solution of the Quadratic by Completing the Square. The method of solution in this case may be observed in the following examples.

EXAMPLE 1. Solve the equation $x^2+27=12\,x$ for x by completing the square.

Transpose the terms containing x and x^2 to the left side of the equation, those not containing an x to the right side of the equation. Thus,

$$x^2-12\,x=-27.$$

To make the left side a perfect square we must take half of 12, *square it, and add it to both sides of the equation.* This gives

$$x^2-12\,x+36=-27+36=9,$$

or

$$(x-6)^2=9,$$

whence

$$x-6=\pm 3.$$

Therefore

$$x=6\pm 3, \quad i.e. \quad x=9 \quad \text{or} \quad x=3.$$

EXAMPLE 2. Solve the equation $7\,x=3\,x^2-6$.

Transposing the unknown terms to one side and the constant to the other side, we find

$$3\,x^2-7\,x=6,$$

or, dividing by 3,

$$x^2-\tfrac{7}{3}\,x=2.$$

Half the coefficient of x is $-\frac{7}{6}$. If we square this and add it to each side, we obtain

$$x^2-\tfrac{7}{3}\,x+\tfrac{49}{36}=2+\tfrac{49}{36}=\tfrac{121}{36},$$

or

$$(x-\tfrac{7}{6})^2=\tfrac{121}{36}.$$

Thus

$$x-\tfrac{7}{6}=\pm\tfrac{11}{6},$$

whence

$$x=\tfrac{7}{6}+\tfrac{11}{6}, \quad \text{or} \quad x=\tfrac{7}{6}-\tfrac{11}{6},$$

i.e.

$$x=3 \quad \text{or} \quad x=-\tfrac{2}{3}.$$

EXERCISES

Solve each of the following equations by completing the square and check the answers.

1. $2\,x^2+4=9\,x$.
2. $3\,x^2+5\,x-12=0$.
3. $4\,z^2-20\,z+25=0$.
4. $8\,p^2-2\,p-15=0$.
5. $5\tan^2 A-8\tan A=21$.
6. $12\sin^2 A+6=17\sin A$.

64. Simultaneous Equations. Two (or more) distinct equations which contain the same unknown quantities are called ***simultaneous equations***, if they are considered at the same time.

Solution of simultaneous equations is possible, in general, only when *the number of given equations is the same as the number of unknown quantities.*

EXAMPLE 1. The cost of a bushel of potatoes and a bushel of apples was \$4.00, but the apples cost 20¢ more than the potatoes. Find the price of each.

Let x = the price of a bushel of apples,

and

y = the price of a bushel of potatoes.

Then

$$x+y=400,$$

and

$$x-y=20.$$

Adding gives

$$2x=420,$$

$$x=\$2.10 \text{ the price of apples,}$$

whence $y=\$1.90$ the price of potatoes.

Either of the above equations taken singly would not be sufficient to determine x and y.

EXAMPLE 2. Solve the simultaneous equations

$$\begin{cases} 2x-5y=60, \\ 7x-12y=342. \end{cases}$$

Multiply the first equation by 7, the second one by 2. Thus

$$14x-35y=420,$$
$$14x-24y=684.$$

Now subtract and we have

$$-11y=-264, \quad \text{or} \quad y=24.$$

For y in the first equation put 24, giving

$$2x-120=60, \quad i.e. \quad x=90.$$

Thus the set of values $x=90$, $y=24$ satisfies the second equation and is said to be *the solution of the set of equations.*

EXERCISES

Solve each of the following sets of simultaneous equations.

1. $\begin{cases} 5a+b=21, \\ 2a+3b=24. \end{cases}$

2. $\begin{cases} 8x+7y=100, \\ 12x-5y=88. \end{cases}$

3. $\begin{cases} 3A+2B=32, \\ 20A-3B=1. \end{cases}$

4. $\begin{cases} \frac{3x}{19}+5y=13, \\ 2x-33=\frac{7y-4}{2}. \end{cases}$

5. The standard daily ration for an adult laboring man should be about 16 oz. starch, 4 oz. fat, and 4 oz. albumen. These materials as found in bread, butter, and beef are as follows.

Food	Starch	Fat	Albumen
Bread	54%	1%	9%
Butter	0%	83%	1%
Beef	0%	15%	15%

Find the quantities of these foods required to make a daily ration.

6. If a dairy cow requires 1 lb. of protein to 6 lb. of carbohydrates in her food, what should be the proportions of dry peas and hay for her daily ration if the dry peas contain 10 lb. of protein and 32 lb. of carbohydrates per bushel of 60 lb., and the hay 88 lb. of protein and 880 lb. of carbohydrates per ton?

65. Fractional Exponents. Thus far we have used only exponents that are positive integers. We defined a^2 to mean $a \cdot a$, $a^n = aaa \cdots$ to n factors, etc.

It is often convenient to use *fractional* exponents, such as $x^{2/3}$. But x taken as a factor $\frac{2}{3}$ times has no meaning. Since it is important that the symbols used should always follow the same rule, we shall, in the case of exponents, extend the definition for combining them so as to include fractional and negative values. That is, we will make the relation

$$a^p \cdot a^q = a^{p+q}$$

always true, whatever p and q may be.

Thus $a^{1/2}$ shall mean a number such that

$$a^{1/2} \cdot a^{1/2} = a.$$

Therefore, $a^{1/2}$ *is the square root of* a and $a^{1/2} = \sqrt{a}$.

Likewise $a^{3/4}$ shall mean that

$$a^{3/4} \cdot a^{3/4} \cdot a^{3/4} \cdot a^{3/4} = a^3,$$

i.e. $a^{3/4}$ *is the fourth root of* a^3 and $a^{3/4} = \sqrt[4]{a^3}$.

Thus, generally, $a^{p/q}$ shall mean a number such that

$$a^{p/q} \cdot a^{p/q} \cdot a^{p/q} \cdots (\textit{to } q \textit{ factors}) = a^p,$$

i.e. $a^{p/q}$ ***is the qth root of*** a^p ***and*** $a^{p/q} = \sqrt[q]{a^p}$.

Therefore, *in a fractional exponent, the numerator denotes a power, the denominator denotes a root.*

EXAMPLES. $a^{1/3} = \sqrt[3]{a}$; $a^{5/6} = \sqrt[6]{a^5} = (\sqrt[6]{a})^5$; $x^{3/2} = \sqrt{x^3}$;

$8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$, or $8^{2/3} = (8^{1/3})^2 = (2)^2 = 4$.

66. Zero Exponent. *To find the meaning of* a^0. From the previous paragraphs

$$a^0 \cdot a^n = a^{0+n} = a^n, \quad \text{whence} \quad a^0 = a^n \div a^n = 1.$$

Thus, *any number whose exponent is zero is equal to 1.*

EXAMPLES. $x^0 = 1$, $10^0 = 1$, $(\frac{3}{8})^0 = 1$, etc.

67. Negative Exponents. *To find the meaning of* a^{-n} (*n positive*). From the previous paragraphs,

$$a^{-n} \cdot a^n = a^0 = 1, \quad \text{whence,} \quad a^{-n} = \frac{1}{a^n}.$$

Therefore, *to change the sign of the exponent of a number, take the reciprocal of the number, and affix the exponent with its sign changed.*

EXAMPLES. $x^{-3} = \frac{1}{x^3}$; $x^{-1/2} = \frac{1}{x^{1/2}} = \frac{1}{\sqrt{x}}$; $\frac{1}{x^{-2}} = x^2$;

$$\frac{1}{x^{-3/2}} = x^{3/2} = \sqrt{x^3}; \quad a + x^{-1} = a + \frac{1}{x}; \quad 8^{-2/3} = \frac{1}{8^{2/3}}.$$

68. Definitions. By definition, therefore,

(1) $a^n = aaa \cdots$ *to n factors if n is a positive integer.*

Thus $a^2 = aa$.

(2) $a^{1/r} = \sqrt[r]{a}$, i.e. *the rth root of a, as* $a^{1/3} = \sqrt[3]{a}$.

$a^{n/r} = \sqrt[r]{a^n}$ = *the rth root of* a^n,

$= (a^{1/r})^n = (\sqrt[r]{a})^n$.

Thus

$$a^{2/3} = \sqrt[3]{a^2} = (\sqrt[3]{a})^2; \quad 25^{3/2} = \sqrt{25^3} = (\sqrt{25})^3 = 125.$$

(3) $a^0 = 1$.

(4) $a^{-n} = \dfrac{1}{a^n}$ (*n positive or negative, fractional or integral*).

69. Laws of Exponents. For n and r any positive or negative number, fractional or integral,

(1) $$a^m \cdot a^r = a^{m+r},$$

$$\frac{a^m}{a^r} = a^{m-r},$$

i.e. *add exponents when multiplying, subtract them when dividing powers of quantities.*

(2) $$(a^n)^r = a^{nr},$$

i.e. *multiply exponents in raising to powers, divide the exponents in extracting roots.*

(3) $$a^n \cdot b^n = (ab)^n.$$

EXERCISES

Write each of the following expressions without zero or negative exponents.

1. a^{-3}. **2.** a^2x^{-1}. **3.** $\dfrac{5\,x^2y^{-1}}{2\,x^{-3}}$. **4.** $\dfrac{a^{-2}+b^{-3}}{2\,a^{-1}b^0}$. **5.** $\dfrac{a^0-x^{-1}}{10^0\,x^{-1}-x^{-2}}$.

6. Evaluate each of the following expressions.

$16^{3/4}$; $(-8)^{2/3}$; $(25^{-1})^{-3/2}$; $\left(\dfrac{9^{-1}}{16}\right)^{1/2}$; $\left(\dfrac{9}{16}\right)^{-1/2}$; $(27^{-1})^{2/3}$.

7. Write each of the following expressions with radicals and positive exponents.

$$ax^{1/3};\ (ax)^{1/3};\ 32^{2/5};\ 32^{-2/5};\ \frac{21\,x^{-3/2}y^2}{3\,x^{-2}y^{-1}};\ (6\,a^2b^{-3})^{2/3}.$$

8. Write each of the following expressions with fractional exponents instead of radicals, and vice versa.

$$\sqrt{3\,ab};\ \sqrt[3]{by};\ \sqrt[3]{x^2a^{-3}};\ \left(\frac{x^2y^{-3}}{x^{-2}y^{-4}}\right)^{3/2};\ x^{2/3}y^{-2/3};\ (5^{-1/2}x^{2/3}b^{3/2})^6.$$

9. Simplify

$$\sqrt{25\,x^2a^4b^{-2}};\ \sqrt[3]{27\,r^3t^6v^2};\ \sqrt{3\,a^2+6\,ab+3\,b^2};\ \sqrt[3]{(x+y)^5(x+y)^{-2/3}}.$$

Perform the indicated operations in each of the following expressions.

10. $(x^{2/3}+y^{2/3})(x^{2/3}-y^{2/3})$.

11. $(a^4+a^2+1)(a^{-4}-a^{-2}+1)$.

12. $(x^{-2/3}+x^{-1/3}+1)(x^{-1/3}-1)$.

13. $(21\,a+a^{2/3}+a^{1/3}+1)\div(3\,a^{1/3}+1)$.

14. $(x^{-3}+2\,x^{-2}-3\,x^{-1})\div(x^{-2}+3\,x^{-1})$.

Add a third term to complete the square in each of the following expressions.

15. $x+2\,x^{1/2}+?$

16. $9+6\,a^{1/2}+?$

17. $a^{-2}+8\,a^{-1}+?$

70. Reduction of Radicals. Similar Radicals. The rules for operating with radicals follow from the preceding paragraphs. Radicals are said to be ***similar*** if they differ only in their coefficients.

A. To introduce a quantity under the radical sign, raise the quantity to the form of the radical and then multiply them together.

EXAMPLES. $5\sqrt{2}=\sqrt{25}\cdot\sqrt{2}=\sqrt{50}.$

$$2\sqrt[3]{2}=\sqrt[3]{8}\cdot\sqrt[3]{2}=\sqrt[3]{16}.$$

B. To reduce a radical to a simpler form, factor the quantity under the radical sign into factors so that the indicated root may be performed on them.

EXAMPLES. $\sqrt{12}=\sqrt{(4)(3)}=\sqrt{4}\sqrt{3}=2\sqrt{3}.$

$\sqrt[3]{54}=\sqrt[3]{27.2}=3\sqrt[3]{2}.$

EXERCISES

Simplify each of the following expressions by taking out a factor from under the radical sign.

1. $\sqrt{288}.$ 2. $\sqrt{980}.$ 3. $\sqrt{750}.$ 4. $\sqrt{49(a+b)^3}.$

71. Addition, Subtraction, Multiplication, Division of Radicals. Similar radicals may be added or subtracted.

EXAMPLE. $2\sqrt{20}+\sqrt{27}-\sqrt{\tfrac{1}{3}}+\sqrt{45}=4\sqrt{5}+3\sqrt{3}-\tfrac{1}{3}\sqrt{3}+3\sqrt{5}$

$=7\sqrt{5}+\tfrac{8}{3}\sqrt{3}.$

Similar radicals may also be multiplied or divided by performing these operations on the coefficients and on the quantities under the radical signs since

$$a^n b^n=(ab)^n \text{ and } \frac{a^n}{b^n}=\left(\frac{a}{b}\right)^n.$$

EXERCISES

Simplify each of the following expressions.

1. $\sqrt{6}+\sqrt{24}-3\sqrt{54}+3\sqrt{6}.$ 2. $\sqrt[4]{\frac{25}{4}};\ \sqrt{\frac{9}{5}};\ \sqrt[6]{\frac{64}{27}};\ \sqrt{1-\frac{1}{x^2}}.$

3. $\sqrt{18}+\sqrt{50}-\sqrt{75}+\sqrt{48}+\sqrt{98}-\sqrt{32}.$

4. $\sqrt{20}\sqrt{45}\sqrt{\tfrac{1}{5}}$ *Ans.* $12\sqrt{5}.$

5. $3\sqrt{54}\sqrt{24}\div\sqrt{150}.$ 7. $\sqrt{12}\sqrt{15}\div\sqrt{24}\sqrt{6}.$

6. $4\sqrt[3]{x^3y^5}\cdot\frac{3}{8}\sqrt[3]{\frac{x^6}{y}}\div 5\sqrt[3]{8\,y^5}.$ 8. $(2\sqrt{3}-4)^2.$

72. Small Errors. Numbers obtained from actual measurements are subject to an error of not more than 5 in the next place after the last one written.

For example, if a measured length is reported to be 3.47′, we understand the true length to be somewhere between

3.465′ and 3.475′. If the last figure is 0, it means that the reading is reliable in that place. For example, 6.20′ means that the true value is somewhere between 6.195′ and 6.205′.

This applies to those measurements which are stated without any indication of the probable error involved.

73. Error in a Sum. Suppose a and b are the readings of two measurements and that e_1 and e_2 are their errors, respectively. The true values are therefore $a \pm e_1$ and $b \pm e_2$. Their sum is therefore

$$(a \pm e_1) + (b \pm e_2) = (a+b) \pm e_1 \pm e_2.$$

The sum of the two readings, $a+b$, may thus have an error of not more than $e_1 + e_2$. This may be expressed by saying:

The error of a sum is the sum of the errors.

EXAMPLE. The sides of a field measure $127' \pm .2'$, $231' \pm .4'$, and $159' \pm .3'$. Find the possible error in the sum of the sides.

$$e_1 = .2',\ e_2 = .4',\ e_3 = .3' \text{ whose sum} = .9'.$$
$$127 + 231 + 159 = 517.$$

The true value of the sum is thus between $517' + .9'$ and $517' - .9'$.

74. Error in a Product. Suppose a and b are two measurements whose errors are e_1 and e_2, respectively. The product of the two magnitudes is then possibly

$$(a \pm e_1)(b \pm e_2) = ab \pm ae_2 \pm be_1 \pm e_1e_2.$$

The error of the product is thus not more than $ae_2 + be_1 + e_1e_2$. But as e_1 and e_2 are both small as compared with a and b, their product e_1e_2 is so small correspondingly that it may be neglected. The error in ab is thus very well represented by

$$ae_2 + be_1.$$

Thus, *the error of a product ab is a times b's error plus b times a's error.*

EXAMPLE. Find the extreme error in the computed area of a rectangular field whose sides are found by measurement to be $(127' \pm .2')$ and $(231' \pm .4')$.

The extreme error made in taking the product 127×231 would be

$$(127)(.4)+(231)(.2)=97 \text{ sq. ft.}$$

The true value of the product of the two given numbers is therefore somewhere between $45{,}507+97$ and $45{,}507-97$, in square feet. It is remarkable that so great an uncertainty in area may result from such small errors in the original measurements.

75. Error in a Quotient. The quotient of two measurements, a and b, whose errors are e_1 and e_2, respectively, is

$$\frac{a \pm e_1}{b \pm e_2} = \frac{a}{b} + \frac{\mp ae_2 \pm be_1}{b(b \pm e_2)}.$$

Since e_2 is small as compared with b, we may replace $b \pm e_2$ in the denominator by b. The error in the quotient is then very well represented by

$$\frac{ae_2 + be_1}{b^2}.$$

Thus, *the error in a quotient* $\frac{a}{b}$ *is a times b's error plus b times a's error, divided by the square of b.*

EXAMPLE. The error in the quotient of $625' \pm .7$ and $36' \pm .2'$ is

$$\frac{(36)(.7)+(625)(.2)}{36^2} = .12$$

The true value of the quotient may therefore be between

$$\tfrac{625}{36}+.12 \text{ and } \tfrac{625}{36}-.12$$

EXERCISES

1. Add (3.61 ± 0.04) and (5.84 ± 0.03) and find the possible error in the sum.

2. What is the extreme error in

$$(55.6 \pm 0.4)+(71.8 \pm 0.5)-(4.3 \pm 0.3)?$$

3. Find the extreme error in the product of (351 ± 0.2) and $(127 \pm .7)$. Test the validity of the answer by multiplying together the extreme possibilities, viz. (351.2) times (127.7) and (350.8) times (126.3).

4. Compute the error in (1672 ± 0.4) times (896 ± 0.5).

5. Compute the errors in the following quotients.

(a) $136 \pm 0.3 \div 52 \pm 0.8$

(b) $526 \pm 0.7 \div 36 \pm 0.4$

(c) $7.321 \pm 0.018 \div 1.234 \pm 0.009$

6. If l is the measured length of the side of a square field and if e is the small error to which it is liable, find the *total* error in the computed area of the field. Also the approximate error.

[HINT. The area is $(l \pm e)^2$.]

7. What is the area and its possible error for a square field measuring 247′ if the measured side is liable to an error of 0.5′?

8. The diameter of a circular plate is 6″. Find the increase in its area due to heating if the radius has thereby been lengthened 0.1″. What does this mean geometrically?

9. What is the error in the volume of a cube whose side is a when the side is liable to an error e in its measurement?

[HINT. The volume is $(a+e)^3 = a^3 + 3\,a^2e + 3\,ae^2 + e^3$. Which of these terms may be neglected? Why? What then is the error in the volume?]

10. The side of a cube measuring 2.1″ on each side is increased 0.02″ due to heating. What is the corresponding increase in volume?

11. Draw a square to represent $(4.01)(4.01)$. Indicate thereon the meaning of 0.01×0.01.

Do the same for a rectangle whose sides are 4.01 and 7.06. Show also the meaning of $(4) \times (0.06)$. Of $(7) \times (0.01)$. Of 4×7.

12. What possible inaccuracy may exist in the area of a rectangle whose measured sides are 22.34′ and 16.47′? Find the area of the rectangle.

[HINT. The error in 22.34 may be as great as 0.005′. Thus the sides may be written $(22.34' \pm 0.005')$ and $(16.47' \pm 0.005')$.]

13. If a bin containing wheat is approximately $6' \times 15' \times 4'$, show that an error of $\frac{1}{2}$ in. in measuring each of the dimensions would make a difference in value of more than \$15.00, if wheat is \$2.20 per bushel.

CHAPTER V

GRAPHIC REPRESENTATION OF QUANTITIES

76. Positive and Negative Numbers. Directed Line Segments. Scales. So far we have considered a line segment, as OX, Fig. 69, to have length only. Its measure has always been a positive number. However, it is often

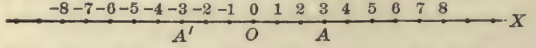


FIG. 69. — POSITIVE AND NEGATIVE DIRECTIONS ON A MARKED LINE

of very great convenience to make use of the notion of positive and negative numbers. This is desirable when we wish to distinguish between the two opposite directions or senses which exist on a line. Let O be any fixed point of reference on a straight line, as in Fig. 69. The position of any other point, as A or A', is known as soon as we know its *distance* from O and its *direction* from O. We may thus say that A is 3 *units to the right of* O and that A' is 3 *units to the left of* O. If we agree that segments measured to the right of O shall be positive and those to the left of O shall be negative, we may say that A is $+3$ units from O and that A' is -3 units from 0.

Such a marked line, showing the positive and negative numbers representing the points in regular order to the right and left of the origin, respectively, is known as a ***number-scale*** or an ***algebraic scale***. A foot rule, or a yardstick, or a centimeter scale are examples of scales. They show only positive numbers. A thermometer is a good illustration of a vertical scale that shows numbers above and below zero, for example, 10° and $-10°$.

The directed segment, used as a unit on scales, is not necessarily a straight line. For example, a protractor is a semicircular scale, § 1, while many greenhouse temperature charts and weather statistics are registered on paper fastened on revolving cylinders.

EXERCISES

1. A bicylist starts from a certain point and rides $+18$ mi. and then -10 mi. on a straight road. How far from and on which side of the starting point is he?

2. How far and in what direction from the starting point is an auto which goes eastward and westward according to the following numbers: $+16$ mi., then -6 mi., then -20 mi., then $+28$ mi., then -15 mi.?

3. Give the meaning of the following latitudes, using the equator as the origin: $+28°$, $-18°$, $+2°$, $-23°$. (What large city in the western hemisphere is in latitude $-23°$? What one in latitude $+23°$?)

How are places located on the earth's surface? Give a comprehensive answer.

4. What do $-4°$, $0°$, $+100°$ mean on a Fahrenheit thermometer? On a Centigrade thermometer?

5. What is the significance of points to the left of the origin when the line represents your bank account?

6. Compare by use of line segments the amount of butterfat in the milk produced by the three types of cows mentioned in Ex. 9, § 8.

77. Graphs of Data. Axes of Reference. We are now enabled to represent graphically various kinds of information implying directly or indirectly positive and negative values. Such representations are used largely in locating points on a diagram, or in locating places as in geography, or in numbering houses in cities. This is done with reference to *two fixed lines or axes*, usually at right angles to each other. In cities two main streets are used as axes of reference, in geography the equator and the prime meridian are so used, while in more local geography township lines, main roads, etc., are assumed as axes of reference.

78. Comparison of Data. This same system may be used to compare two sets of quantities or data where *a value in one set depends upon or corresponds to a value in the other set.* Examples of frequent occurrence are time of day and the temperature in degrees, the age of a calf and its weight, the distance traveled by an auto and the number of hours since starting. Information of this kind can be exhibited on a diagram that shows the pairs of corresponding values. To do this, ***squared paper***, or ***rectangular coordinate paper***, is usually used. This is merely paper ruled into small squares or into small rectangles. Two scales or axes at right angles are chosen — one for each set of data to be compared. *Pairs of related values of the data* are represented by points which are located with reference to these two axes. By connecting these points taken in order, a ***curve*** is obtained showing the different values, their changes from one value to another, etc.

EXAMPLE. On a winter's day the thermometer was read at 6 A.M. and every hour afterward until 5 P.M. The readings were: $-10°$, $-8°$, $-7°$, $-5°$, $0°$, $+2°$, $+8°$, $+8°$, $+10°$, $+5°$, $0°$, $-5°$. Plot this on squared paper and obtain the temperature line.

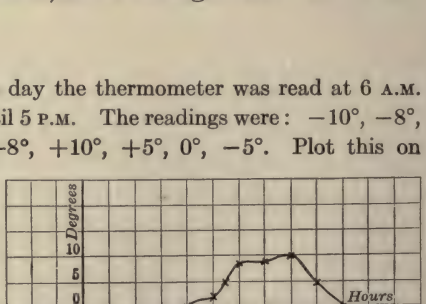


FIG. 70. — TIME–TEMPERATURE GRAPH

Two lines at right angles are chosen as axes, Fig. 70. Time in hours is measured on the horizontal axis, with one large space to the hour. The temperature in degrees is measured vertically upward and downward with one small space to each degree. Thus for 9 A.M. we count 3 spaces to the right and 5 spaces down, locating a point. In this same way we obtain a set of points for all of the data. By joining these points the curve is obtained.

From this temperature line we may obtain various kinds of information, *e.g.*: When was it warmest? When coldest? When was the

temperature *changing most rapidly?* When was the change a rise? What way would you suggest of distinguishing between a *rising change* and a *falling change?*

79. The Graph. Such a diagram as that of § 78 is called a ***graph.*** It furnishes a practical means of studying the relations between the two sets of quantities involved (time of day and temperature in this case). New information may be obtained from the graph.

EXERCISES

1. The rainfall in inches at Buffalo, N. Y., on the six days following April 6, 1908, was .46, .02, .20, .14, 0, .14. Represent this graphically.

2. Using, in any city or town familiar to you, any two streets that run at right angles to each other as axes, locate several points of interest in the city.

3. The following figures give the temperature at intervals of three hours of a typhoid fever patient in a New York Hospital, beginning at 11 P.M. and running several days: 102.2, 102, 101.5, 101.4, 99.6, 101.5, 101.8, 102, 102, 102.4, 101.3, 101, 99.6, 100.6, 102.7, 102.3, 102.5, 101.5, 101, 101.7, 99.8, 100.5, 102.4, 102.2, 102.2, 101, 101, 100.4, 99.3, 100, 101.5, 102.5, 102.2, 101.3, 100.7, 100.4, 100. Trace the rise and fall of temperature by means of a curve. This curve is characteristic of typhoid.

4. The *daily gain* in *weight* of a calf in pounds for periods of one hundred days is given in the following table.

Age in days	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
Daily gain in pounds	3.2	2.8	2.55	2.3	2.16	2	1.9	1.8	1.7	1.63	1.57	1.5	1.47

Draw a curve showing this information. Days may be plotted on the horizontal scale and pounds on the vertical scale. When is the gain most rapid?

5. The relation between the *age* of a calf and the *cost* in dollars per 100 pounds gain in weight is given in the following table:

Age in days	0	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400	1500
Cost per 100 lb. gain in weight	$3.40	3.83	4.25	4.86	5.40	6.08	6.75	7.50	8.45	9.55	10.70	12.00	13.35	14.70	16.10	17.85

Draw the graph for this information.

6. The losses from hog cholera per 1000 of hogs in the United States for the years ending April 1, 1894, to 1917 inclusive, are given in the following table.

Year	1894	1895	1896	1897	1898	1899	1900	1901	1902	1903	1904	1905	1906	1907	1908	1909	1910	1911	1912	1913	1914	1915	1916	1917
Losses	44	80	115	129	83	75	58	67	46	52	52	47	46	43	48	47	45	44	80	100	105	85	60	44

Show in graphic form the annual losses from hog cholera throughout this term of years.

NOTE. Hog cholera serum came into use about 1907. Examine the curve with this fact in mind but remember also that during the years when the curve was ascending little was known regarding the effectiveness of hog cholera serum, that its use was but imperfectly understood, and that during these years the supply of serum was far short of the demand.

7. The lengths of a thousand ears of corn were measured and arranged in half-inch groups as follows:

Length in inches . . .	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0	10.5	11.0	11.5
Number in each group	5	6	13	17	18	55	61	73	80	98	113	134	142	100	53	26	5	1

Plot these measurements and draw the graph, using lengths on the horizontal scale and the number of ears of any given length on the vertical scale.

From the graph what would seem to be an "average" length of these thousand ears of corn? In Chapter X it will be shown how to find this average more definitely.

8. The temperatures as observed on a certain day in January, 1918, at Columbus, Ohio, were the following.

Midnight − 7° F.	9 A.M. − 9° F.	5 P.M. +16° F.
1 A.M. − 8	10 A.M. 0	6 P.M. +12
2 A.M. − 9	11 A.M. + 9	7 P.M. + 8
3 A.M. −10	12 A.M. +15	8 P.M. + 2
4 A.M. −12	1 P.M. +18	9 P.M. + 1
5 A.M. −15	2 P.M. +20	10 P.M. 0
6 A.M. −16	3 P.M. +21	11 P.M. − 2
7 A.M. −16	4 P.M. +20	Midnight − 6
8 A.M. −15		

Draw the graph of this information on rectangular coördinate paper. How long did it take the temperature to rise from its lowest to its highest point? What was the *average rate in degrees* per hour during this time? When was the temperature rising more rapidly than this average rate?

9. Due to the curvature of the earth, objects on the earth's surface are visible only from corresponding elevations above the earth's surface. From the approximate rule*

$$\text{Elevation (in feet)} = \tfrac{2}{3}\ (\text{Distance in miles})^2,$$

the following table of corresponding values is obtained.

Elevation in feet	1	5	10	20	30	40	50	100	150	200	300	500	1000	5000	10,000	20,000
Distance in miles	1.23	2.74	3.87	5.48	6.71	7.75	8.66	12.3	15	17.3	21.2	27.4	38.7	86.6	123	173

Plot the graph of these relations. Would there be any advantage in using a scale for elevations from 1 to 100 ft. different from that used for the other elevations? Explain.

From the graph, how far can a man see if he is 6 ft. tall? From the rule, how far can he see?

10. Of 100,000 persons born alive at the same time, the following table shows the infant mortality during the months of the first year.

Month	Deaths	Month	Deaths	Month	Deaths
1	4377	5	705	9	492
2	1131	6	635	10	456
3	943	7	579	11	421
4	801	8	533	12	389

Draw the graph for these data, plotting number of deaths on the vertical scale.

11. Draw, on the same diagram, graphs of the following time table, using distances on the vertical scale and time on the horizontal scale. Mark the name and the location of each station on the vertical scale. In one case Columbus is the origin, in the other Zanesville is taken as origin.

* For an explanation of this rule, see § 160.

Miles	Stations	Time	Time	Miles	Stations	Time	Time
0	Columbus .	5:55	7:55	0	Zanesville .	5:55	7:55
11	Reynoldsburg	6:33	8:38	13	Nashport . .	6:28	8:28
15	Wagram . .	6:40	8:45	18	Blackhand .	6:37	8:37
18	Etna . . .	6:47	8:50	21	Hanover . .	6:44	8:44
22	Kirkersville .	6:55	8:58	22	Clay Lick .	6:46	8:46
28	Hebron . .	7:05	9:05	28	Newark . .	7:05	9:05
37	Newark . .	7:30	9:30	28	Newark . .	7:10	9:10
37	Newark . .	7:35	9:35	37	Hebron . .	7:36	9:36
43	Clay Lick .	7:55	9:55	42	Kirkersville .	7:48	9:48
44	Hanover . .	7:57	9:57	47	Etna . . .	7:55	9:55
47	Black Hand .	8:04	10:04	50	Wagram . .	8:01	10:01
51	Nashport . .	8:12	10:12	54	Reynoldsburg	8:08	10:08
64	Zanesville .	8:45	10:45	64	Columbus .	8:45	10:45

If the railroad is single track, where and at what time should the trains on the above schedule pass?

12. Draw a graph which is *slowly* increasing. One which is *rapidly* increasing. Draw slowly and rapidly *decreasing* graphs.

13. Draw a graph which is *increasing*, but which is increasing *more rapidly* from point to point along the curve. One which is increasing but *less rapidly* from point to point.

14. Draw a graph which is decreasing, but *more rapidly* decreasing from point to point. Decreasing, but less rapidly.

Which of the above cases would be illustrated by the amount of snow collected at each turn in rolling up a snowball?

Is the phrase, "The rich grow richer, the poor, poorer," an example? Would a tax system based on the rule that the more property one owns the higher the rate, be an example?

Would the application of fertilizers to the growing of crops be an illustration of any of these cases of more rapidly increasing graphs?

15. The following table shows the height in inches of a castor bean plant on different days.

Days	0	20	40	60	80	100	120	140	150	160
Height	0	8	24	34	42	54	72	110	130	145

Draw the graph and from it find the rate of growth. When was the rate of growth most rapid, slowest?

80. Leveling. The art of determining and representing the inequalities in the surface of any portion of the earth is called ***leveling***. The principle is to find the elevation of each point above some assumed ***base*** or ***datum***.

If a line of sight be made, by any means, to revolve about a vertical line to which it is perpendicular, a horizontal plane is described. If a rod, graduated from the bottom up, be held at any point on the ground, it will be cut by this horizontal plane at a distance above ground equal to the height of the line of sight above this point on the ground. This distance from the bottom of the rod, indicated by its graduations, is called the ***reading of the rod***. If the elevation of the line of sight is known with reference to the assumed base or datum, we can find, by subtracting the rod reading from this elevation, the elevation of the point on the ground.

The horizontal line of sight is secured by means of an instrument designed for the purpose called a ***level***.

For example, if the rod reading at A, Fig. 71, is 2.73′ and at B it is 12.15′, and if the line of sight is 100′ above the datum, then the elevation of A above the datum is 97.27′

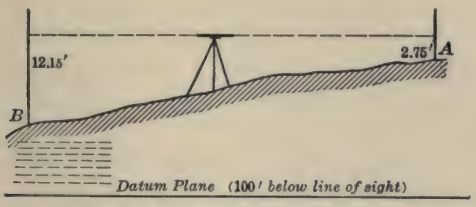


FIG. 71.—LINE OF SIGHT AND DATUM PLANE IN LEVELING

and of B is 87.85′. Conversely, if the elevation of the point is given, then *the elevation of the line of sight above the datum* is found by adding the rod reading to the elevation of A.

Again, if the elevation of one point above a base or datum is known, then the elevation of a second point may be found

by applying the difference in elevation of the two points to the known elevation. This may be repeated at each station. Ordinarily, stations are 50′ or 100′ apart.

A diagram or graph of the elevations showing in detail the rise and fall of the surface over which the course passes, *i.e.* a *vertical section* of the course, is called a ***profile*** of it. The profile is made by plotting the elevations of the points to scale at the proper horizontal distances apart and connecting these plotted points by a line. ***Profile paper*** is used for this purpose. It is ruled with *small divisions between horizontal lines* and *large divisions between the vertical lines*. The vertical distances are exaggerated since they are small as compared with the horizontal distances.

EXERCISES

1. The following level readings in feet were taken at stations 100 feet apart along Grafton Road:

Station No.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Elevation	19.9	18.3	16.2	15	15.3	16	16.7	16.2	14.5	13.3	10.4	10	13.7	17	16.2	17.4	17.8	18

Make a profile of these readings using 100 ft. to the inch for the horizontal scale and 20 ft. to the inch vertically.

If a traction line were to be laid on this road, certain cuts and fills would be made. Sketch such a proposed line on the profile and mark the amount of cut or fill at each station.

2. A farmer runs a level line for a ditch. He first takes the height of a stump in the vicinity of the ditch and then the elevation every 100 ft. along the line of the ditch, driving stakes at these places. The measurements are as follows above datum. Top of stump 40 in.; at stake no. 1, 60 in.; at no. 2, etc., 36 in.; 48 in.; 42 in.; 39 in.; 36 in.; 40 in.; 30 in.; 48 in.; 24 in.; 27 in.; 36 in.; 41 in.; 28 in.; 22 in.; 12 in. Make a profile of these measurements using, as near as is convenient, the same scale as in the previous exercise. Next establish the grade, marking the amount to be taken out at each stake so that the ditch will be 24 in. deep near the stump and slope gradually to 6 in. deep at the last stake.

3. The following elevations in feet, with the bench mark (B.M.) 100′ above datum, were taken at stations 100′ apart along the route of a proposed ditch.

Sta. No.	Back Sight	Height of Line of Instrument	Rod Reading	Elevation of Ground	Fall of Drain	Elevation of Drain	Depth of Drain
B.M.	—	—	—	100.00	—	—	—
	5.32	105.32	—	—	—	—	—
0	—	—	9.43	95.89	Outlet	—	—
0	—	—	5.82	99.50	Surface	95.89	3.61
1	—	—	5.54	99.78	.35	96.24	
2	—	—	5.02	100.30	.35		
3	—	—	4.60	100.72	.35		
4	—	—	5.07	100.25	.35		
5	—	—	3.10	102.22	.35		
—	8.41	110.63	—	—	—	—	—
6	—	—	7.84	102.79	.90		
7	—	—	7.07	103.60	.90		
8	—	—	6.43	104.20	.90		
9	—	—	5.63	105.00	.90		
10	—	—	5.23	105.40	.90		
11	—	—	4.38	106.25	.90		
12	—	—	3.93	106.70	.90		
13	—	—	3.39	107.24	.90		
—	6.06	113.30	—	—	—	—	—
14	—	—	5.30	108.00	.50		
15	—	—	5.00	108.30	.50		
16	—	—	4.70	108.60	.50		
17	—	—	4.05	109.25	.50		
18	—	—	3.90	109.40	.50		
19	—	—	3.57	109.75	.50		
20	—	—	3.13	110.17	.50		

First. Make a profile of the surface elevations at each station.

Second. Calculate and record the elevations of the drain at each station if from stations 0 to 5 the rise per 100′ is .35′, from stations 5 to 13 it is .90′ per 100′, and from stations 13 to 20 it is .50′ per 100′.

Third. Calculate and record the depth of the drain at each station.

81. Price Curves. Graphs may be constructed and used for determining, without computation, costs of different quantities of goods, interest on money, etc. The following example will illustrate the process.

EXAMPLE. If lard sells at 35¢ per pound, the relation between the number of pounds and the cost is expressed by the equation,

$$\text{Cost} = 35 \text{ times the number of pounds, } i.e.$$

$$y = 35\,x,$$

where x is the number of pounds and y is their cost. If values are given to x, corresponding values may be found for y as in the following table.

If $x =$	0	2	4	7	10
then $y =$	0	70	140	245	350

On rectangular coördinate paper select two lines at right angles to each other as axes and call their point of intersection, 0, the ***origin***, Fig. 72. On the vertical axis, OY, let a space represent 10¢ and on the horizontal axis, OX, let a space represent one pound. Thus 0 pounds and 0 cents corresponds to the origin; 2 pounds, 70 cents determines a point 2 spaces to the right of the origin and 7 spaces up. This is written (2,70). Likewise 4 pounds, 140 cents, *i.e.* (4,140) determines a point 4 spaces to the right and 14 spaces up. Also 7 pounds, 245 cents, *i.e.* (7,245) and (10,350) determine points. These points lie on a straight line. It is called the ***price curve***. Price curves are straight lines in many cases.

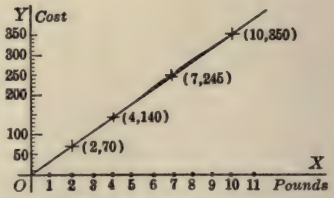


FIG. 72 — POUNDS–COST GRAPH

By looking at this graph or curve, the cost of any number of pounds, even of a fractional number of pounds, may be determined. Thus to get the cost of 8 pounds, find the point where a vertical line 8 spaces to the right meets the price curve; the point horizontally opposite this on the axis OY is the cost. This is 28 spaces from 0 and hence represents 280 cents. Similarly, 9½ pounds cost 333 cents.

EXERCISES

1. Since the price curve in the Example solved in § 81 is a straight line, how many of the points would be essential to draw it? Should these be taken close together or far apart in order to get the position of the price curve most exact?

2. Draw the price curve for coffee at 38¢ per pound. From this curve find the cost of $8\frac{1}{2}$ pounds; 17 pounds; 24 pounds.

3. Draw the price curve for sugar at $11\frac{1}{2}$¢ per pound. By use of this curve, find the cost of 16 pounds; 42 pounds; $8\frac{1}{2}$ pounds.

4. A single railway ticket is to cost \$2.50. If 10 tickets be purchased, the cost will be \$2.25 each. If 50 be purchased, the cost will be \$1.80 each; if 100, the cost will be \$1.50 each; and for 200, it will be \$1.25 each.

Draw a graph for this information. Use it to find the probable cost per ticket for a party of 75. For a party of 125.

About how many tickets must be used to reduce the expense per head to \$2.00? To \$1.60?

5. A certain kind of desk is quoted at \$30.00 each. If one dozen are ordered, they will cost \$28.50 each. For 6 dozens, the price is \$22.50 each, and for 150 the price is \$20.00 each. Draw a graph of these data, and from it estimate the cost of a single desk when 36 are ordered. What, when 100 are ordered?

82. Other Methods of Comparing Quantities. So far the statistical data which we have studied were exhibited on a graph or curve drawn with reference to two rectangular axes. There are various other methods and devices for representing quantities. Some of these are as follows.

(1) ***Line Segments including Bar Diagrams.***

(2) ***Area Charts*** such as

(*a*) *Squares and Rectangles.*

(*b*) *Sectors of Circles.*

(*c*) *Dots.*

(*d*) *Differently Shaded Areas*, as on maps of portions of the earth's surface.

83. Line Segments. The first example given below will illustrate a variation in the use of the graph. The second example illustrates the use of ***bar diagrams.***

EXAMPLE. 1. According to government regulations, the cost of domestic money orders is given by the following graph.

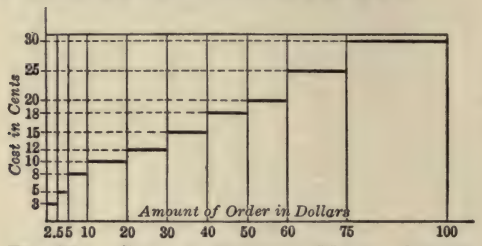


FIG. 73. — COST OF DOMESTIC MONEY ORDERS

EXAMPLE 2. The following table gives the areas before 1914 of some European countries before the Great War, and of some of the

Country .	Russia	Texas	Germany	France	California	Gt. Britain	Ohio
Sq. Miles .	2,095,616	265,780	208,830	207,054	158,360	121,027	41,060

states of the United States. The adjacent figure, Fig. 74, represents them by line segments or so-called bar diagrams.

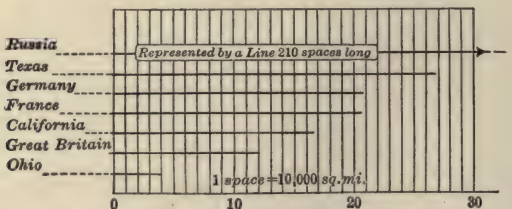


FIG. 74. — AREAS BY BAR DIAGRAMS

A great many different variations of this method are in common use. Thus the number of fruit trees in various states may be represented by pictures of actual trees, whose heights are proportional to the number of trees in the states they represent. It should be stated whether the given data are represented by the heights or by the volumes of the objects pictured.

EXERCISES

1. Parcel post rates in effect Jan. 1, 1914, are given in the following table. Exhibit this in graphic form.

Weight of Parcel	Rates		
	1st and 2d Zones	3d Zone	4th Zone
Not over 4 oz.	1 cent per ounce or fraction thereof		
Over 4 oz. up to 1 lb. . .	5¢	6¢	7¢
Over 1 lb. to 2 lb. . . .	6¢	8¢	11¢
For each pound over 2 lb. up to 20 lb.	1¢ additional	2¢ additional	4¢ additional

2. Make a diagram of segments showing the periods of incubation of poultry as given in the following table.

Name of Fowl	Days	Name of Fowl	Days	Name of Fowl	Days
Common Hen .	21	Pea Fowl	28	Partridge . .	24
Pheasant . .	25	Guinea .	25	Duck, Barbary	30
Duck, common	28	Goose . .	30	Turkey . . .	28

3. Show in graphic form the production of wheat in 1918 in the United States and in some other leading wheat-producing countries as given in the following table.

Region	Bushels	Acreage
United States	917,000,000	59,100,000
Australia	152,400,000	11,500,000
Canada	210,300,000	17,350,000
France	147,150,000	10,260,000
Germany	90,140,000	3,590,000
Argentina	187,210,000	17,175,000

84. Area Charts. In example 2, § 83, the table of areas given there was shown on a bar diagram. That table of areas will now be shown in various ways as follows.

A. By a series of squares.

The sides of the different squares are to each other as the square roots of the respective areas. Thus the side of the

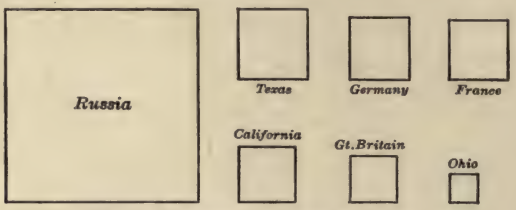


FIG. 75. — COMPARING AREAS BY SQUARES

square which represents the area of Russia is about three times that for Texas, etc. The squares are then constructed to a convenient scale. Is the comparison as effective in this form as in the bar diagram? Why?

B. By sectors of circles.

Inspection shows the area of Russia to be about twice that of all the others in this table combined. It should therefore

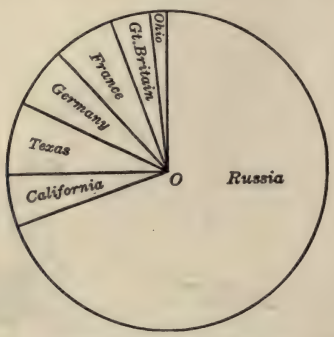


FIG. 76. — COMPARING AREAS BY SECTORS OF CIRCLES

occupy $\frac{2}{3}$ of the circle. Texas is about $\frac{1}{4}$ the remainder, and is so represented on the circle.

C. By use of dot diagrams.

FIG. 76 *a*. — DOT DIAGRAM

D. By differently shaded areas. This is best illustrated by reference to ordinary maps of these countries, made to the same circle.

EXERCISES

1. Exhibit in graphic form the following table of fertilizing materials and the per cent of each element carried.

MATERIAL	% NITROGEN	% PHOSPHORIC ACID	% POTASH
Ground bone	3.0	22.0	
Dissolved bone . . .	3.0	15.0	
Cottonseed meal . . .	7.0	2.5	1.5
Tankage	6.0	11.0	
Fish scrap	8.0	7.0	
Wood ashes		1.5	5.0

2. Make a dot diagram of the world's wheat production for 1914 as given by the following data.

The World	3,362,299,000 bushels.
United States	891,017,000 bushels.
Russia	776,960,000 bushels.
France	319,667,000 bushels.
Hungary (proper)	105,237,000 bushels.
India (British)	312,032,000 bushels.

CHAPTER VI

GRAPHS IN ALGEBRA

85. The Equation and its Graph. We have seen so far that various relations between two quantities, *as these quantities change*, may be represented by graphs. In some cases we knew the law of this change as represented by an equation. For example, for lard at 35¢ per pound, the law was :

Cost = 35 times the number of pounds.

The graphs were constructed by using two axes at right angles to each other. On these axes we used the number scales for various quantities such as cost, pounds, time, distance, etc. In general then we may use y and x to represent these quantities and thus describe the relations between them in the language of algebra. This leads very often to an equation in y and x. Thus for lard at 30¢ per pound, the cost, y, of any number of pounds, x, is

$$y = 35\,x.$$

86. Rectangular Coördinates. It is thus evident that when *the relation between two quantities is expressible by an equation, this relation may be exhibited by a corresponding graph.* The following considerations are commonly accepted as the *basis* for such representation. Let us call the ***horizontal axis or scale, the x-axis***, and the ***vertical axis, the y-axis***. There being no restriction on these scales as to length, we give x and y each positive and negative values.

Let the two axes be drawn at right angles to each other. Their point of intersection at O is called the ***origin***, Fig. 77. They divide the plane into four parts, called ***quadrants***, named as shown in the figure.

From the origin to the *right* and from the origin *up* is regarded as *positive* in direction, but to the *left*, also *down*, is *negative*. We can now locate definitely any point in the plane (of course with reference to these two fixed lines). Thus the notation (2, 3) means the point located by going 2 units to the right on the x-axis and then from this place

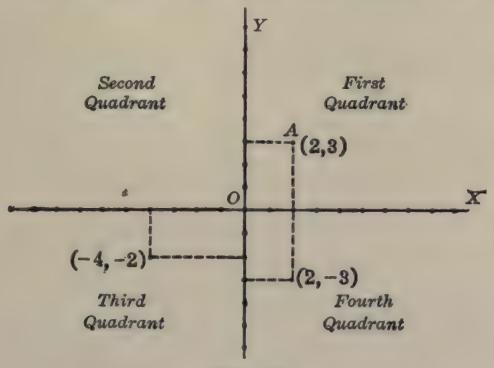


FIG. 77

3 units *up* parallel to the y-axis, as shown in Fig. 77. Likewise, $(2, -3)$ means to go 2 units to the right on the x-axis and then 3 units *down* parallel to the y-axis; $(-4, -2)$ means 4 units to the left and 2 units down. The distance from the vertical axis is commonly called the ***abscissa*** of the point, and the distance from the x-axis is known as the ***ordinate*** of the point. These two together are called the coördinates of the point. The notation for the coördinates of a point is thus very precise, viz. *written in parenthesis with the x-coördinate or abscissa first and separated from the y-coördinate by a comma.*

EXERCISES

1. Locate the points (1, 5), (3, 7), (5, 2), (9, −3), and (12, −6). Draw lines connecting these points in the order given.

2. Draw the graph determined by connecting by a smooth curve the points (−3, −2), (−1, 0), (0, 1), (2, $\frac{1}{2}$), (5, 7), and (8, −11).

3. Let (0, 0) and (4, 0) be two vertices of an equilateral triangle. Find the coördinates of the third vertex in the first quadrant. What is the area of this triangle? What acute angles are involved in this triangle?

4. Find the area of the triangle whose vertices are the points (2, 3), (5, 6), (4, 7), by dropping perpendiculars from each vertex on the y-axis.

87. Linear Equations and Their Graphs. *Every equation of the first degree represents a straight line.* We saw above that many price curves are straight lines. This is due to the fact that the cost is equal to the price of one article times the number of articles, *i.e.*

$$y = px, \tag{1}$$

where y is the cost of x articles at p units each. But this equation is equivalent to the ordinary proportion,

$$\frac{y}{p} = \frac{x}{1}. \tag{2}$$

Now in Fig. 78, AE represents the cost, p, of one article, since from (1) $y=p$ *when* $x=1$. BD represents the cost of two articles, LC that of x or any number of articles.

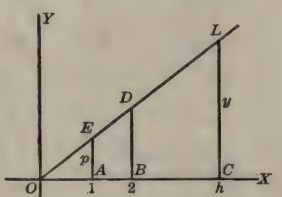


FIG. 78. — LINEAR GRAPH

The triangles OAE, OBD, OCL are similar since

$$\frac{CL \text{ (or } y)}{AE \text{ (or } p)} = \frac{OC \text{ (or } x)}{OA \text{ (or } 1)}.$$

But this similarity of triangles can hold only when OL is a straight line, and the proportion (2) holds only when the equation (1) is of the first degree.

Therefore an equation of the first degree represents a straight line.

State the converse of this proposition, and show that it is true from Fig. 78.

88. Graph of an Equation. Let us construct the graph for the equation

$$y = 2x + 4.$$

We first give a value to x, as $x = 0$, and find the *corresponding* value of y. In this case $y = 4$. For $x = 1$, y is 6, etc., as in the following table.

x	y
0	4
1	6
2	8
3	10
−1	2
−2	0
−3	− 2
−8	−12

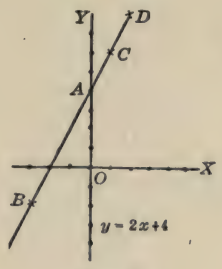


FIG. 79

Now locate, as in Fig. 79, the points whose coördinates are these pairs of x and y values. Thus $x = 0$, $y = 4$, *i.e.* (0,4) locates the point A on the diagram; $(-3, -2)$ locates B. It appears that the graph is a straight line. Why is it a straight line?

EXERCISES

Find several points on the graph of each of the following equations, and draw the graph.

1. $y=x$, $y+x=0$.
2. $y+x-2=0$.
3. $y=x+3$.
4. $2x-3y=6$.
5. $x+y+1=0$.
6. $\frac{x}{2}+\frac{y}{3}=1$.
7. $-\frac{x}{4}+\frac{y}{6}=1$.
8. $\frac{x}{3}-y=1$.
9. $y-1=3(x-2)$.

10. What is the *degree* in x and y of each of the equations just plotted? What kind of graph does each represent? What is the reason for calling these equations *linear* equations?

11. Compare the straight lines $y=x$ and $y=x+3$. If x be given the same value in each equation, how are the corresponding y-values related? Try several values, say $x=1$, $x=2$, $x=4$. Thus the values $x=1$, $y=1$, fix a point on the first line; the values $x=1$, $y=4$, fix a point on the second line. These y-values differ by 3. Try other values of x.

12. What happens to the equation when a straight line, say $y=x+1$, is raised three units parallel to itself? Raised five units? Lowered four units?

13. Plot $y=2x$. This is the equation of a straight line through the origin, since for $x=0$, y is 0. What is the equation of a straight line parallel to $y=2x$ and 3 units above it? How much larger is every y-value?

14. Suppose that the cost of setting the type in a job of printing is \$1.50, and the cost of paper and printing is $\frac{1}{2}$¢ per copy. Find the cost, y, of any number of copies, x (*i.e.* $y=$ total cost and $x=$ any number of copies).

Draw the price curve from the cost equation.

[HINT. In plotting the equation, let one vertical small space equal 10¢ and one horizontal small space equal 10 copies.]

15. The cost of setting the type for an order of printing is \$1.00. The cost of material and printing is $\frac{1}{5}$¢ for each copy. What is the cost for any number of copies? Draw the graph. Estimate from the graph the price of 150 impressions. Of 500 copies.

16. For the job of printing in Ex. 15, another firm offers to disregard the initial cost of typesetting, and to print the copies for $\frac{1}{2}¢$ each. Find the cost for any number of copies. Draw the graph on the same diagram as for the previous example.

17. Of the two firms in the Exs. 15–16, which one will most cheaply furnish 100 copies? Which firm will give the greater number of copies for \$1.25? For \$2.00? How many copies will be printed just as cheaply by one firm as by the other? What will they cost?

18. The initiation fee in a certain society is \$5.00 and the annual dues are \$3.00. What is the cost of membership for any number of years? Draw the graph. Estimate from this graph the cost of membership for 5 years. For 12 years.

19. The cost of mimeographing is quoted by a firm as follows: 25 copies of 1 page, 50¢; 50 copies of 1 page, 57¢; 100 copies, 70¢; 200 copies, 95¢; 300 copies, 120¢; 400 copies, 145¢; 500 copies, 170¢. Plot this data, and draw the price curve. What is the graph?

20. Draw a graph showing the relation between the Fahrenheit and Centigrade thermometers.

Plot the Fahrenheit scale as the vertical axis and the Centigrade scale as the horizontal axis. The Fahrenheit scale is marked 32° at the freezing point, while 0° is the freezing point on the Centigrade scale. Hence 32° must be subtracted from Fahrenheit readings before comparing with the Centigrade reading. Find the difference between the readings for freezing and for boiling on each scale. What is the ratio between these two differences? Establish the algebraic relation, and plot the graph from it. *Ans.* $F° = 9/5\, C° + 32$.

21. Calculate algebraically the Centigrade reading for 41° F., 0° F., −20° F., +98° F., +68° F. Do your answers correspond to the graphical results?

In the same way calculate, graphically and algebraically, the Fahrenheit readings for the following Centigrade readings: 14° C., 25° C., 20° C., −10° C.

89. Two Points Determine the Graph of a Straight Line. We have seen that every equation of the first degree in two variables represents a straight line. Since a straight line is fixed by any two points on it, to find the graph for an

equation of the first degree, it is only necessary to find two points on the graph and draw the straight line through them.

EXAMPLE. Plot the graph of the equation $3x-4y+12=0$.

For $x=2$, y is $4\frac{1}{2}$, and for $x=-1$, y is $2\frac{1}{4}$. Locate the points $(2, 4\frac{1}{2})$, $(-1, 2\frac{1}{4})$, and draw the straight line as in Fig. 80.

90. Intercepts. Since $x=0$ for every point on the y-axis, to find where any curve *crosses* the y-axis, we *put* $x=0$, and *find* y. Similarly to find the x-intercept, *put* $y=0$ and *find* x. In the example just given, viz. $3x-4y+12=0$, if $x=0$, y is 3, and when $y=0$, x is -4. It is seen in Fig. 80 that the straight line passes through the points $(-4, 0)$ and $(0, 3)$ on the axes. Such distances cut off on the axes by a curve or by a straight line are called the ***intercepts***.

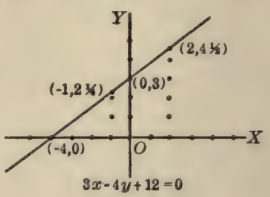


FIG. 80

91. Simultaneous Solution of Equations. In solving simultaneous equations, for example, the pair

$$\begin{cases} 2x-y=1, \\ 3x+2y=12, \end{cases}$$

we find the pairs of values of x and y which satisfy *both* equations. To solve the given pair, multiply the first one by 2 and add the result to the second equation. This gives

$$7x=14, \text{ or } x=2.$$

In either equation, if $x=2$, $y=3$. Thus $(2, 3)$ is the set of values that satisfies *both* equations.

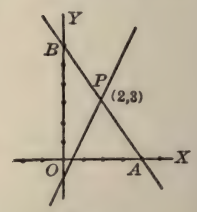


FIG. 81

Let us now see what this means with reference to the graphs of these equations. Plot them by finding their intercepts, as in Fig. 81. Each line passes

through the point (2, 3). Hence the coördinates for the point of intersection of two graphs is the set of values obtained by solving them simultaneously.

EXERCISES

Draw the graphs for each of the following pairs of equations. Find their solution from this diagram, and determine whether this agrees with their algebraic solution.

1. $\begin{cases} 2x-3y=25, \\ x+y=5. \end{cases}$ **3.** $\begin{cases} 5x+2y=8, \\ 2x-3y=-12. \end{cases}$ **5.** $\begin{cases} \frac{x}{3}-\frac{y}{5}=-1, \\ -\frac{x}{4}+\frac{y}{2}=1. \end{cases}$

2. $\begin{cases} 5x+6y=7, \\ 2x-4y=-4. \end{cases}$ **4.** $\begin{cases} 2x-3y=6, \\ \frac{x}{2}+\frac{y}{4}=1. \end{cases}$ **6.** $\begin{cases} 3-4x=y, \\ 8x+2y=6. \end{cases}$

7. The equations of the sides of a triangle are

$$x-y+1=0, \qquad 2x+y=13, \qquad x+5y=11.$$

Plot these equations and find each of the vertices of the triangle.

8. Find the length of each of the sides of the triangle of Ex. 7.

[HINT. Use each side of the triangle as the hypotenuse of certain right triangles formed by drawing lines through the vertices parallel to the axes.]

9. A rectangular field is 32 rods longer than it is wide. The length of the fence around it is 308 rods. Find the dimensions of the field.

10. A steamer makes 6 mi. an hour against the current and 19½ miles an hour with the current. What is the rate of the current and what is the steamer's rate in still water?

11. The difference between two sides of a rectangular wheatfield is 30 rods. A farmer cuts a strip 5 rods wide around the field and finds the area of this strip to be 7½ acres. Find the dimensions of the field.

12. Given two boys A and B, a 30-lb. weight, and a teeter board. They find they will balance when B is 6 ft., and A 5 ft., from the fulcrum; but if B places the 30-lb. weight on the board beside him, they will balance when B is 4 ft., and A 5 ft., from the fulcrum. Find the weight of each boy, using the law of levers; that *the weights balanced are inversely proportional to their respective distances from the fulcrum.*

92. Sine, Cosine, Tangent of an Obtuse Angle. Let XOB, Fig. 82, be an obtuse angle. From a point on OB drop a perpendicular, BC on XO produced. OC is then negative, § 76, and OB is regarded as positive. The sine, cosine, tangent of XOB are now defined by the following ratios:

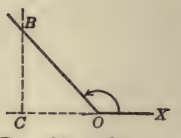


FIG. 82. — OBTUSE ANGLE

$$\sin XOB = \frac{CB}{OB}, \text{ and is } +,$$

$$\cos XOB = \frac{OC}{OB}, \text{ and is } -,$$

$$\tan XOB = \frac{CB}{OC}, \text{ and is } -.$$

It follows that *the sine of the obtuse angle XOB is identically equal to the sine of its supplementary acute angle COB;* and that *the cosine and the tangent of the obtuse angle XOB are numerically equal, respectively, to the cosine and the tangent of its supplementary acute angle COB, but opposite in sign.*

Hence, if A is any obtuse angle and $180° - A$ its supplement, we have

(1) $\sin (180 - A) = \sin A.$

(2) $\cos (180 - A) = -\cos A.$

(3) $\tan (180 - A) = -\tan A.$

EXAMPLES.

$\sin 150° = \sin (180° - 150°) = \sin 30° = +.5$

$\cos 150° = -\cos (180° - 150°) = -\cos 30° = -.8660$

$\tan 150° = -\tan (180° - 150°) = -\tan 30° = -.5774$

$\sin 110° = \sin (180° - 110°) = \sin 70° = .9397$

$\cos 110° = -\cos (180° - 110°) = -\cos 70° = -.3420$

$\tan 110° = -\tan (180° - 110°) = -\tan 70° = -2.747$

93. Reading of Tables for Obtuse Angles. To find from the table the value of the sine, cosine, or tangent of an obtuse angle, *find the value of the same function of the supplement of the angle*, and for the cosine and the tangent prefix the negative sign.

EXAMPLES. $\sin 112° 10' = +\sin 67° 50' = +.9261$
$\cos 112° 10' = -\cos 67° 50' = -.3773$
$\tan 112° 10' = -\tan 67° 50' = -2.4545$

Conversely, if $\cos A$ or $\tan A$ is negative, find from the table the angle whose cosine or tangent is respectively this same positive value. Then subtract this angle from 180°. The result is A.

EXAMPLES. Find A, if $\cos A = -.8192$

Look for $+.8192$ in the column of cosines. This corresponds to 35°. Then $A = 180 - 35° = 145°$.

Again, if $\tan A = -.5774$, find A.

Look for $+.5774$ in the column of tangents. This corresponds to 30°. Then $A = 180° - 30° = 150°$.

If $\sin A$ has a given positive value, there are two angles which have this sine, one an acute angle and the other its supplement.

EXERCISES

1. Given $\sin A = 3/5$, where A is an obtuse angle. Find $\cos A$, $\tan A$.

2. Given $\tan A = -5/12$, where A is an obtuse angle. Find $\sin A$, $\cos A$.

3. A flagstaff 80 feet high, on a horizontal plane, casts a shadow 110 feet long. Find the angle made by the sun's rays with the horizontal.

Find the value of the following expressions from a table of natural functions.

4. $\sin 131° 10'$ **6.** $\tan 120°$ **8.** $\tan 95° 15'$

5. $\cos 150° 20'$ **7.** $\cos 95° 15'$ **9.** $\sin 95° 15'$

Find A in each of the following expressions.

10. $\sin A = .4746$ **12.** $\cos A = -.4746$ **14.** $\tan A = -.4125$

11. $\cos A = -.8090$ **13.** $\tan A = -1.7321$ **15.** $\cos A = -.5819$

94. Grade, Slope. In an earlier section, § 4, we spoke of the *pitch* of a roof. It was defined as the rise of the roof divided by the width of the building. Applied to a hill, we saw that the rise of the hill divided by its run is its *slope* or its *grade*. Mention was made also of the fact that many grades, particularly in railroad construction, are spoken of in terms of per cent; for example, a 5% grade is one that rises 5′ in 100′.

A little later, § 36, this notion was made more precise in studying the right triangle. *The rise of the hypotenuse of a right triangle divided by its run*, i.e. *the altitude divided by the base*, was used as a measure of the base angle, and was called the *tangent* of this angle.

The same notion of slope will be used in what follows for any straight line.

95. Slope of a Straight Line. Let a straight line pass through two points $P=(x_1, y_1)$ and $Q=(x_2, y_2)$, Fig. 83. Drop perpendiculars parallel to the axes as shown. Then

$$\text{slope of } PQ = \frac{AQ}{PA} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Thus *the slope of a line between two points is equal to the difference of the y-coördinates of the points divided by the difference of their x-coördinates, subtracted in the same order.*

In Fig. 83 *a*, the slope of PQ is *positive*, since it is the rise divided by the run, in moving along the line to the right. Conversely, a rise divided by a run means a positive slope.

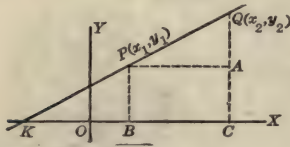


FIG. 83 *a*. — POSITIVE SLOPE

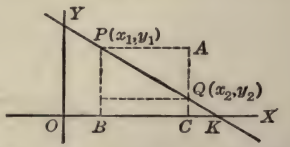


FIG. 83 *b*. — NEGATIVE SLOPE

This corresponds to the *tangent of the acute angle* XKP which this line makes with the x-axis on the right side of the line.

But, in Fig. 83 *b*, the slope of PQ is *negative*, since it is the fall divided by the run, in moving to the right along the line. Conversely, fall divided by run means a negative slope. This corresponds to the *tangent* of the obtuse angle XKP which this line makes with the x-axis on the right side of the line, § 92.

EXAMPLE 1. Find the slope of the line through the two points $P=(3, 4)$ and $Q=(8, 6)$. Find also the angle which this line makes with the x-axis, as angle XKP, Fig. 83 *a*.

SOLUTION: Taking the two points in the order P, Q, we find

$$\text{Slope of } PQ = \frac{4-6}{3-8} = \frac{-2}{-5} = +.4,$$

or, if the points are taken in the order Q, P, we have

$$= \frac{6-4}{8-3} = +.4,$$

so that the answer is the same in either case. Hence

$$\tan XKP = .4,$$

whence

$$\angle XKP = 21° 48'.$$

EXAMPLE 2. Find the slope of the line through the two points $P=(3, 7)$ and $Q=(10, 2)$. Find also the angle which this line makes with the x-axis on the right side of the line.

SOLUTION:

$$\text{Slope of } PQ = \frac{7-2}{3-10} = -\frac{5}{7} = -.7143,$$

or

$$\tan XKP = -.7143, \quad \text{and} \quad \angle XKP = 180° - 31° 12' = 148° 48'.$$

EXERCISES

1. Construct a line through (0, 0) whose slope is 3/4.

[HINT. A slope $+3/4$ means a rise of 3 and a run of 4. Therefore begin at (0, 0), rise 3 units and run 4 units to the right. Connect the final point with (0, 0). The resulting line will have the slope 3/4.]

2. Construct a line through (0, 0) whose slope is $-3/4$.

[HINT. A negative slope means a fall and a run. Therefore, fall 3 units from (0, 0) and run 4 units to the right.]

3. Construct a slope of $5/7$ from the point (1, 2). Also one of $-5/7$ from the same point.

4. Compare the slopes $3/4$ and $-4/3$ from the same point (2, 2). Do you recognize the angle between these slopes?

5. Compare, as in exercise 4, the slopes $-2/3$ and $+3/2$, from the same point (3, 4).

For each of the following pairs of points:

(*a*) Plot the points.
(*b*) Draw the straight line through them.
(*c*) Find the slope of the straight line.
(*d*) Find the angle the line makes with the x-axis.

6. (1, 2) and (3, 3).
7. (1, 2) and (3, −1).
8. (3, 2) and (−3, −4).
9. (−3, 2) and (2, −4).
10. (−3, −4) and (−1, −1).
11. (−3, −4) and (−5, 1).

96. To Find the Equation of a Straight Line. Heretofore we have had certain equations given and our problem was to find the graphs of these equations. Our problem now is *to find the equation when the graph is given.* The problem here is to connect in some algebraic way the relation between the x-distance and the y-distance of a point which will hold for all points on the line.

For example, if a point is located *anywhere on the x-axis* the one thing we can say of this point is that its y-coördinate is always zero. The concise algebraic statement of this fact is the equation

$$y = 0.$$

Hence this is the equation of the x-axis, since it is the one statement that is true for any and all points on the x-axis, and for no other point.

What is the equation of the y-axis?

Again, let us find the equation of the line which bisects the first and third quadrants and passes through the origin. Let (x, y), Fig. 84, be *any* point on this line. The problem is to find some relation between the y-distance and the x-distance which will be true for all points on this line. Since $POA = 45°$, it follows at once that PA or y is equal to OA or x. This fact is stated algebraically by the equation

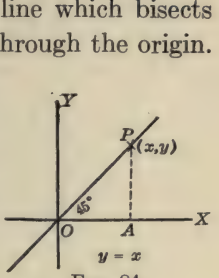


FIG. 84

$$y = x, \quad \text{or} \quad x - y = 0.$$

97. Line through Two Points. Line through One Point in a Given Direction. *A straight line is fixed definitely in position if it passes through two fixed points,* or *if it passes through one fixed point in a given direction.* The equation of the line in either case may be found by using the slope of the line. In one case the slope between the two given points may be found, in the other case it is given. The method will be shown by some particular examples.

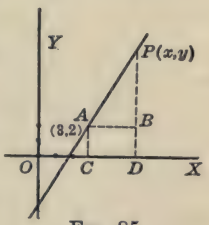


FIG. 85

EXAMPLE 1. Find the equation of the straight line which passes through the point (3, 2) and has the slope 2.

SOLUTION: Plot the point (3, 2), Fig. 85. Draw a line through this point with the slope 2. Let $P(x, y)$ be any point on the line. The slope from A to P is

$$\frac{BP}{AB} = 2.$$

But

$$BP = PD - AC = y - 2, \quad \text{and} \quad AB = OD - OC = x - 3.$$

It follows that

$$\frac{y-2}{x-3} = 2.$$

Hence this is the equation of the line. Clearing of fractions, we find

$$y = 2x - 4.$$

Attention is called to the *form* of this last equation, in which the slope of the line appears as the coefficient of x.

EXAMPLE 2. Find the equation of the straight line through (1, 2) and (5, 4).

SOLUTION: Plot these points and draw the line through them. Now take *any* point R on the line with coördinates (x, y), Fig. 86. Then

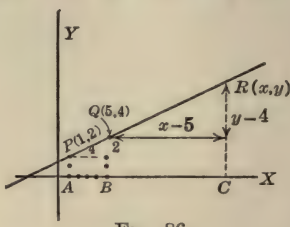


FIG. 86

$$\text{slope } P \text{ to } Q = \text{slope } Q \text{ to } R,$$

or $$\frac{4-2}{5-1} = \frac{y-4}{x-5},$$

or $$\frac{1}{2} = \frac{y-4}{x-5},$$

from which $$y = \tfrac{1}{2}x + \tfrac{3}{2}.$$

What is the slope of the straight line? Does the equation of the straight line show this? Note the form of this equation.

EXERCISES

1. Find the equations of the straight lines passing through the following points and having the given slopes.

(*a*) Through (3, 4) with the slope 2/3.
(*b*) Through (3, 4) with the slope −2/3.
(*c*) Through (−3, 4) with the slope 6/5.
(*d*) Through (2, −3) with the slope −3.

2. What is the equation of a line parallel to the x-axis and four units above it? Four units below it?

What is the equation of a line parallel to the y-axis and twenty units to the right of it? Parallel to the y-axis and three units to the left?

3. What is the slope of a straight line bisecting the first quadrant? Does this same line (extended) bisect the third quadrant? Why?

Find the equation of the straight line bisecting the second and fourth quadrants. What is its slope?

4. Find the equation of a line through each of the following pairs of points. Reduce each equation to the form showing its slope. Find the x-intercept and the y-intercept in each case.

(*a*) (3, 4) and (−2, 2).
(*b*) (3, 2) and (5, 6).
(*c*) (−6, 1) and (−1, −5).
(*d*) (2, 4) and (1, −1).
(*e*) (−15, −3) and (8/3, −7/9).
(*f*) (−1, 0) and (4, −2).

5. Write the equation of each of the straight lines described below.

(*a*) A line whose x-intercept is 4 and whose y-intercept is 7, *i.e.* a line through (4, 0) and (0, 7).

(*b*) A line whose x-intercept is -3 and whose slope is 4/5.

(*c*) A line whose y-intercept is 8/3 and whose slope is -2.

6. Find the equation of a line through the origin (0, 0) and the point (5, 3). Of course there are no intercepts in this case. How does this fact appear in the equation?

7. Find the equation of a straight line whose x-intercept is twice as great as its y-intercept and which also passes through the point $(-2, 3)$.

[HINT. What is the slope of the line?]

8. Find the equation of the straight line which makes equal intercepts on the axes and which passes through the point $(2, -4)$.

9. Find the equation of the straight line which makes an angle of $60°$ with the x-axis and which passes through the point $(2, -3)$.

[HINT. What function of the angle is the slope?]

10. A ditch rises 1 ft. in 50 ft. Draw a diagram of it and find its equation.

11. Find all the points for which $x=2$, *i.e.* draw the line whose equation is $x=2$. Do the same for $x=-2, 0, 1, 3, -3$.

12. Draw the lines $y=0, 1, -1, 2, 3, -4$, respectively.

98. Parallel Lines. Draw two non-vertical parallel lines as shown in Fig. 87. Select any two points on each line. The slopes are, respectively,

$$\frac{a_1}{b_1} \text{ and } \frac{a_2}{b_2}.$$

The triangles having these sides are similar. Hence

$$\frac{a_1}{b_1} = \frac{a_2}{b_2},$$

and the two slopes are equal. *Therefore, if two non-vertical lines are parallel, their slopes are equal.* State and prove the converse.

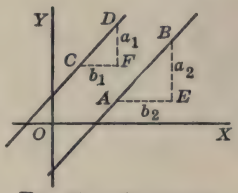


FIG. 87. — PARALLEL LINES

99. Perpendicular Lines. Draw two non-vertical, perpendicular lines, Fig. 88.

The slope of the one line is

$$m_1 = -\frac{a_1}{b_1}. \quad \text{(Why negative ?)}$$

The slope of the other line is

$$m_2 = \frac{a_2}{b_2}.$$

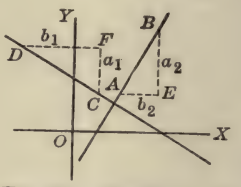


FIG. 88. — PERPENDICULAR LINES

But, as the triangles are similar,

$$\frac{a_2}{b_2} = \frac{b_1}{a_1} = -\left(\frac{1}{-\frac{a_1}{b_1}}\right),$$

i.e.

$$m_2 = -\frac{1}{m_1}.$$

Hence, if two non-vertical lines are perpendicular to each other, the slope of one is the negative reciprocal of the slope of the other.

EXERCISES

1. A line passes through the points (2, 5) and (6, −1). What is its slope? What is the slope of a line perpendicular to it?

Ans. −3/2; 2/3.

2. What is the slope of the line whose equation is

$$y = \tfrac{3}{4}x + 2?$$

Is the point (4, 5) on this line? What is the slope of a line perpendicular to it?

3. Discuss the slopes of the following equations.

(*a*) $y = 2x + 3$, (*c*) $y = -\tfrac{1}{2}x + 5$,

(*b*) $y = 2x + 4$, (*d*) $y = -\tfrac{1}{2}x + 6$.

4. Find the equation of a line through the point (1, −2) parallel to $y = 2x + 3$.

[HINT. From the given equation, what is the slope? Having found the slope, you can find the equation as required.]

5. Find the slope and the intercepts on the axes for each of the following lines, and draw the lines.

(*a*) $3x-4y=12$. (*c*) $x+y=5$. (*e*) $\frac{x}{2}+\frac{y}{3}=1$.

(*b*) $5x+2y=10$. (*d*) $6x+3y=2$.

6. Give the slopes of lines perpendicular to the lines whose equations are given in Ex. 5.

100. Motion Problems Involving Time, Rate, Distance. Many problems involving the element of time may be solved graphically. If a man walks 6 miles in 2 hours at a steady rate, he walks at the rate of 3 mi. per hour. This may be represented on a diagram, Fig. 89, by plotting time (in hours) on the horizontal axis and distance (in miles) on the vertical axis according to the relation

distance = 3 times the number of hours traveled,

or, what is the same thing,

$$y=3x,$$

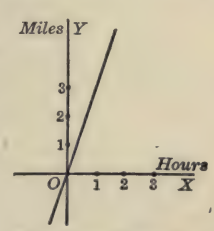


FIG. 89.—TIME-DISTANCE GRAPH

if y means *distance* traveled and x means *hours* traveled. This gives the straight line OP, which represents geometrically the uniform motion of the man.

EXAMPLE. An automobile starts from Columbus, going 30 miles per hour. Two hours later a motorcyclist starts in pursuit at the rate of 54 miles per hour. If they continue at these rates, when and how far from the starting point will the motorcycle overtake the automobile?

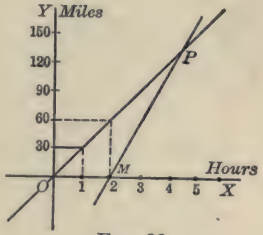


FIG. 90

In the diagram, Fig. 90, let each large horizontal space represent 1 hr., and each large vertical space 30 mi. Then in 1 hr. the auto goes 30 mi., in 2 hr. 60 mi., etc. The points representing this, *i.e.* (1 hr. 30 mi.), (2 hr. 60 mi.), etc., lie on a straight line OP.

The motorcyclist starts at zero miles after 2 hours. This is represented by the point M. He goes 54 mi. in 1 hr. Hence his motion is repre-

sented by the graph MP. The point where these two graphs intersect shows the place and time of meeting. From the diagram the time is $4\frac{1}{2}$ hr. and the distance is 135 mi. Solve the problem algebraically and thus check the accuracy of the graphical solution.

Of course, the graphical solution is not always exact. In most cases it is only approximate, but in any case it serves as a check and a guide to the algebraic solution. However, *all measurements are only approximations*, — accurate only to a certain degree. In the preceding problem, the rate of 30 mi. per hour is only an approximation. Thus our graphical, approximate solutions are frequently as accurate and reliable as the data from which these solutions are made.

For example, in transferring degrees Fahrenheit to degrees Centigrade by the graphical process the approximate answer is often desirable. In fact very few thermometers except those made especially for scientific purposes are really accurate. Few thermometers give the correct temperature to within half a degree. In no case should an answer be expressed more accurately than the circumstances warrant.

EXERCISES

[Solve the following problems graphically.]

1. A starts for a town 12 mi. distant, walking at the rate of 3 mi. per hour. An hour and a half later B starts for the same place, driving at the rate of $7\frac{1}{2}$ mi. per hour. When does B overtake A? Where is A when B reaches the town?

2. In a mile race, A runs 6 yd. per second and B 5 yd. per second. If B starts 250 yd. ahead of A, who will win the race and how far ahead is the winner at the finish?

[HINT. Plot one small vertical space equal to 20 yd. and one small horizontal space equal to 5 sec.]

3. At 6 A.M. a freight train leaves a station O going north at the rate of 30 mi. per hour. At 9 A.M. an express train is due at this same

station coming from the opposite direction at 60 mi. per hour. When and where shall these trains be ordered to pass?

4. How long will it take a steamboat, going at the rate of 8 mi. per hour, to overtake a motor boat 10 mi. off, going directly away at the rate of 5 mi. per hour?

5. A train traveling 30 mi. per hour is followed 3 hr. after its start by a train making 50 mi. per hour. When and where will the fast train be ordered to pass the first one?

6. Kansas City and Chicago are 498 mi. apart. Where do two trains pass, one starting from Chicago at 40 mi. per hour at the same time another leaves Kansas City at 50 mi. per hour?

7. A man rows downstream at the rate of 6 mi. per hour and rows back at the rate of 3 mi. per hour. How far can he go and return in 9 hours?

[HINT. Construct a line representing the downward journey at 6 mi. per hour. Then, beginning at the point which represents 9 hr., construct the points which represent his position at each *preceding* hour, *i.e.* count back 1 hr. and up 3 mi. The intersection of these lines is the result required.]

8. A man goes from Chicago to Milwaukee on a train making $42\frac{1}{2}$ mi. per hour and returns at once on a steamer making 17 mi. per hour. Find the distance between the cities if the trip requires 7 hr.

9. A freight train leaves a station at 12 M. At 1:04 P.M. a fast train making 40 mi. per hour leaves the same station in the same direction and is ordered to pass the freight at 2:40 P.M. What is the rate of the freight train, and how far is the point of passing from the station?

10. A man rides a wheel into the country at the rate of 8 mi. per hour. After riding a while an accident occurs and he walks back at the rate of 3 mi. per hour, reaching home 11 hr. after the start. How far did he go?

CHAPTER VII

COMPUTATION BY LOGARITHMS

101. Introduction. Computation is ordinarily carried on by the processes of multiplication, division, raising to powers, and extraction of roots. These arithmetic processes are often laborious by the ordinary methods. They may be greatly abridged by the use of tables of logarithms of numbers. Logarithms of the trigonometric ratios are useful in trigonometric problems. The following table illustrates the principles involved.

102. Table of Exponents. The following table of exponents gives the first eight successive powers of 10. The table enables us to reduce multiplication and division of these numbers to addition and subtraction, respectively, due to the first law of exponents:

$$a^p \; a^q = a^{p+q}.$$

Table of Exponents of 10

$10^0 = 1.$	$10^3 = 1000.$	$10^6 = 1{,}000{,}000.$
$10^1 = 10.$	$10^4 = 10{,}000.$	$10^7 = 10{,}000{,}000.$
$10^2 = 100.$	$10^5 = 100{,}000.$	$10^8 = 100{,}000{,}000.$

103. Extension of the Table of Exponents. It occurs at once that a more complete table of exponents would be quite useful in performing multiplications and divisions. The table of § 102 may be extended to contain numbers less than 1 by simply dividing 1 by 10 repeatedly and sub-

tracting 1 from the exponent each time. This conforms precisely to the way this table of exponents was made. Thus we find

$10^2=100,\ 10^1=10,\ 10^0=1,\ 10^{-1}=\frac{1}{10},\ 10^{-2}=\frac{1}{100}$, etc.

The gaps in such a table may be filled in by taking arithmetic means of successive exponents and geometric means, § 144, of the corresponding numbers. For example, the arithmetic mean between 2 and 3 is 2.5. The number corresponding to this in the second column of the table of exponents must then be $\sqrt{(100)(1000)}$, *i.e.*

$$\sqrt{100{,}000}=100\sqrt{10}=316.2,$$

which means that $(10)^{2.5}=316.2$ In practice, however, these tables are computed by a more rapid process, which is too advanced to be discussed here.

104. Definition of a Logarithm. The exponents of 10 in the first column of the table of § 102 are called the ***logarithms*** of the numbers in the second column, and 10 is called the ***base*** of the system.

In general, *the logarithm of a number is the exponent by which the base is affected to produce the given number.* Thus, if n is the number, x the exponent, and b the base, then

$$b^x=n \quad \text{and} \quad x=\log_b n$$

are equivalent expressions, where $\log_b n$ is read, *the logarithm of n to the base b.*

For example, $3^2=9$ is the equivalent of $\log_3 9=2$; $4^3=64$ is the equivalent of $\log_4 64=3$. If $2^x=32$, then $x=\log_2 32$, *i.e.* $x=5$.

105. Logarithms to the Base 10. Any positive number except 1 might be used as a base for a system of logarithms. The system with 10 as base is most generally used; it is called the ***common system of logarithms.***

In this book, when no base is given, 10 will be understood to be the base; and log N will mean $\log_{10} N$.

In general, logarithms are not integers but are composed of an integer and a fraction. The fractional part is usually written as a decimal fraction. Thus, since

$$10^2 = 100, \quad i.e. \quad \log 100 = 2,$$

and

$$10^3 = 1000, \quad i.e. \quad \log 1000 = 3,$$

the logarithm of any number between 100 and 1000 is 2 plus a certain decimal. What is wanted is a means of finding this decimal part in each instance. These decimal parts have been determined and *tabulated.* Such a tabulation is called a ***table of logarithms.*** (See § 108.)

106. Characteristic. Mantissa. The integral part of a logarithm is called the ***characteristic,*** and the decimal part is known as the ***mantissa.***

Mantissas are found from a table, characteristics are determined by inspection. Considerations of the tables of §§ 102–103 lead to the following rule.

Rule for Characteristic. *The characteristic of logarithms of numbers greater than unity is one less than the number of digits to the left of the decimal point.*

For example,

$$\log 324 = 2.5105, \quad \log 32.4 = 1.5105, \quad \log 3.24 = 0.5105$$

For positive numbers less than unity, the characteristic of the logarithm is negative but numerically one greater than the number of ciphers immediately following the decimal point.

For example,

$\log .00324 = -3$ and $+.5105$, written $7.5105-10$, since $7-10=-3$,

$\log .0324 = -2$ and $+.5105$, written $8.5105-10$.

$\log .324 = -1$ and $+.5105$, written $9.5105-10$.

Observe that while some characteristics are positive and some are negative, *the mantissas are always positive.*

Dividing a number by 10 moves the decimal point one place to the left, and subtracts *unity* from the logarithm of the number. This does not affect the mantissa, but the characteristic is decreased by unity.

Similarly, multiplying a number by 10 moves the decimal point one place to the right, and adds unity to the logarithm of the number. The mantissa remains unchanged, but the characteristic is increased by unity. For example, if (from a table)

$$\log 376 = 2.5752,$$

then $\log 37.6 = 1.5752$, $\log 3.76 = 0.5752$,
$\log .376 = 9.5752 - 10$, $\log .0376 = 8.5752 - 10$, etc.

107. Properties of Logarithms. Since a logarithm is, by definition, an exponent, the rules for exponents apply to logarithms. Thus, since

$$10^a \cdot 10^b = 10^{a+b},$$
$$10^a \div 10^b = 10^{a-b},$$
$$(10^a)^p = 10^{ap},$$
$$(10^a)^{1/p} = 10^{a/p},$$

we have the following rules.

I. *The logarithm of the product of two numbers is equal to the sum of the logarithms of the factors of this product,* i.e.

$$\log MN = \log M + \log N.$$

For example, $\log 6 = \log 2 + \log 3$ since $6 = 2$ times 3.

II. *The logarithm of the quotient of two numbers is equal to the logarithm of the numerator minus the logarithm of the denominator,* i.e.

$$\log \frac{M}{N} = \log M - \log N.$$

For example, $\log (4/7) = \log 4 - \log 7$.

III. *The logarithm of the pth power of a number is equal to p times the logarithm of the number*, i.e.

$$\log (N^p) = p \log N.$$

For example, $\log (2^3) = 3 \log 2$.

IV. *The logarithm of the rth root of a number is found by dividing the logarithm of the number by r*, i.e.

$$\log \sqrt[r]{N} = \frac{1}{r} \log N.$$

For example, $\log (7^{1/2}) = \frac{1}{2} \log 7$.

EXERCISES

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 5 = .6990$, find the value of each of the following expressions.

1. $\log 4$.
2. $\log 6$.
3. $\log 8$.
4. $\log 9$.
5. $\log 10$.
6. $\log 12$.
7. $\log 15$.
8. $\log \sqrt{5}$.
9. $\log \sqrt{8}$.
10. $\log 8/9$.
11. $\log \sqrt{\frac{8}{25}}$.
12. $\log \left(\frac{8^3 \cdot 5^2}{3^4 \cdot 6}\right)^{1/3}$.

108. The Table of Logarithms. It is supposed here that a table of mantissas to four decimal places is used. This meets the requirements of most of the calculating to be done in agriculture, and much of the calculating to be done in engineering.

From a table of logarithms we can find:

(*a*) *the logarithm of a given number,*

(*b*) *the number corresponding to a given logarithm.*

The general principles governing the use of tables may be explained by the following examples. The tables are found in the Appendix of this book, Table VIII.

EXAMPLE 1. Find the logarithm of 374.

The characteristic is 2. Now look in the table in the left-hand column, headed N at the top, for 37. Then follow horizontally across to the mantissa in the column headed 4 at the top, where 5729 is found. This is the required mantissa. Thus $\log 374 = 2.5729$

EXAMPLE 2. Find log 37.

For log 37, we look for 37 in the column headed N. Opposite this, in the column headed O is the mantissa .5682 Thus log $37 = 1.5682$

EXAMPLE 3. Find log 32.73

The characteristic is 1. This number cannot be found in our table, but we can obtain the mantissa for its logarithm by a process called ***interpolation.*** This assumes that *to a small change in the number there corresponds a proportional change in the mantissa.* The number 3273 lies 3/10 of the way from 3270 to 3280. Hence the mantissa for log 32.73 is similarly placed between log 3270 and log 3280. To obtain log 32.73, we find from the table

Mantissa of log 3270	= .5145
Mantissa of log 3280	= .5159
Difference	= 14

Our desired mantissa is therefore .5145 + 3/10 of 14 = .5149 Hence we have log $32.73 = 1.5149$

This may be arranged schematically as follows:

	Number	*Mantissa*	
Difference = 10	3270	5145	$x = 4$
	3273		14 = difference,
	3280	5159	

from which $x : 14 = 3 : 10$, *i.e.* $x = 3/10$ of $14 = 4$.

EXAMPLE 4. Find N if $\log N = 0.8485$

This mantissa cannot be found in our table, but we find the mantissas next below and above it, viz. 8482 and 8488. These mantissas correspond to 7050 and 7060. Thus since 8485 is 3/6 of the way from 8482 to 8488 we conclude that its corresponding number is 3/6 of the way from 7050 to 7060, *i.e.* 7055. Since the characteristic is O the number N is 7.055

Schematically this may be arranged as follows:

	Number	*Mantissa*	
	7050	8482	
Difference = 10 x			3
		8485	6 = difference,
	7060	8488	

from which $x : 10 = 3 : 6$, *i.e.* $x = 5$. This added to 7050 gives N.

EXERCISES

1. Find from the table the logarithms of each of the following numbers.

(*a*) 47.3 (*c*) (.063)(417) (*e*) $(.0349)^{1/3}$
(*b*) .0473 (*d*) $(.0349)^{1/2}$ (*f*) $(.00349)^{3/4}$

[HINT. To find $\log (.0349)^{1/2}$, *i.e.* $\frac{1}{2} \log (.0349)$, take $\frac{1}{2}(8.5428-10)$, *i.e.* $\frac{1}{2}(18.5428-20)$. This is equal to 9.2714 To divide by 3, write it $28.5428-30$.]

2. Find the logarithms of each of the following numbers.

(*a*) 243.6

[HINT. Find first log 243.0 The tabular difference or mantissa difference is 18. The correction for the fourth figure 6 is then $\frac{6}{10}$ of $18=11$. This added to log 243.0 is the desired logarithm. Notice also that 11 is found in the *6th column at the right* in the same row as 24.]

(*b*) 3.874 (*c*) .8358 (*d*) 68.72 (*e*) 4657.

3. Find the numbers corresponding to each of the following logarithms.

(*a*) 1.4335 *Ans.* 27.13 (*d*) 3.4775
(*b*) 1.4341 *Ans.* 27.17 (*e*) $9.5687-10$
(*c*) $8.4341-10$ *Ans.* .02717 (*f*) 8.7741

109. Computation by Logarithms. The principal use of logarithms is in shortening computation. The outline of the work in the following examples suggests similar arrangements for other exercises. It is desirable to make first the outline of the work to be done, then look up all logarithms and enter them in their proper places. After this has been done, the necessary additions, subtractions, etc. are performed. This procedure saves time, and reduces very much the chances of error.

EXAMPLE 1. Find 25.66×18.43

(1) $\log 25.66 = 1.4092$
(2) $\log 18.43 = 1.2655$
Sum of (1) and (2) $= 2.6747$
$= \log 472.8$
Hence, the product $= 472.8$

EXAMPLE 2. Find the value of $\dfrac{124.8 \times .03843}{.7687 \times 49.56}$.

(1) $\log 124.8 = 2.0962$

(2) $\log .03843 = 8.5847 - 10$

(3) Add (1) and (2) $= 10.6809 - 10$

(4) $\log .7687 = 9.8858 - 10$

(5) $\log 49.56 = 1.6951$

(6) Add (4) and (5) $= 11.5809 - 10 = 1.5809$

(7) Subtract (6) from (3) $= 9.1000 - 10$

$= \log (.1259)$

Hence, the value $= .1259$

EXAMPLE 3. Find the value of $x = \dfrac{(.3748)^{1/3}(29.36)^2}{\sqrt{431.5}}$.

$\log x = \frac{1}{3} \log .3748 + 2 \log 29.36 - \frac{1}{2} \log 431.5$

(1) $\log .3748 = 9.5738 - 10 = 29.5738 - 30$ (in order to divide by 3)

(2) $\frac{1}{3} \log .3748 = 9.8579 - 10$

(3) $\log 29.36 = 1.4677$

(4) $2 \log 29.36 = 2.9354$

(5) Add (2) and (4) $= 12.7933 - 10 = 2.7933$

(6) $\log 431.5 = 2.6350$

(7) $\frac{1}{2} \log 431.5 = 1.3175$

(8) Subtract (7) from (5) $= 11.4758 - 10$

$= 1.4758$

$= \log 29.91$

Hence, $x = 29.91$

MISCELLANEOUS EXERCISES

Find the value of each of the following expressions by means of logarithms.

1. (*a*) $\dfrac{(2348)(.4537)^3}{(83.46)}$. (*c*) $\sqrt[3]{92.83}$

(*b*) $\sqrt{13.84}$ (*d*) $\sqrt{\dfrac{3547}{(1.045)^7}}$.

2. (*a*) $\dfrac{\sqrt{3979}\sqrt[4]{6854}}{\sqrt[3]{627.8}}$. (*b*) $\left(\dfrac{4481}{73.96}\right)^{2/5}$.

3. The amount of a principal at compound interest for a certain time is given by the formula

$$A = P(1+r)^n$$

in which A = the amount, P = the principal, r = the rate, n = the number of years or periods.

Find by use of this formula, the amount of \$598.40 for 15 years at 5% interest compounded annually. Compounded semiannually. Compounded quarterly.

[HINT. For compounding semiannually, use n = twice the number of years and r = half the rate. For compounding quarterly, use n = four times the number of years and r = one fourth the rate.]

4. A field whose successive vertices are A, B, C, D, E is measured as follows: $AB = 21$ rods, $BC = 39$ rods, $CD = 23$ rods, $DE = 30$ rods, $EA = 18$ rods, $BE = 35$ rods, $BD = 38$ rods.

Draw the figure to scale, divide it into triangles, and compute the area of each triangle in acres from the formula for the area in acres of a triangle,

$$A = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{160},$$

where a, b, c are, respectively, the sides of the triangle, and $s = \frac{1}{2}(a+b+c)$.

5. What is the capacity in barrels (40 gallons) of a round cistern 12 feet deep and 5 feet in diameter?

6. The time of one vibration (one complete swing) of a pendulum is given by the formula

$$T = \pi\sqrt{l/g},$$

where l = length of the pendulum,
g = attraction of gravitation,
= 32.16 ft. = 980.2 centimeters.

Find the time of vibration of a pendulum 21.56″ long. Of one 62.8 centimeters long.

Find the length of a seconds pendulum, *i.e.* one which makes one vibration in one second.

7. How long is a pendulum that vibrates 3 times in 1 second? How long is it if it vibrates 4 times in 3 seconds? What is the time of vibration of a pendulum 40 ft. long?

8. A clock having a seconds pendulum gains 2 minutes per day. How much too short is the pendulum?

9. The pendulum of a certain clock is of brass and is 36″ long at 70° F. A brass rod will expand or contract approximately .00001052 of its length for each Fahrenheit degree change in temperature. What is the length of this pendulum at 40° F.? How much will the clock gain in 1 week due to this change of temperature?

10. A certain concrete sidewalk 578 feet long is laid in a region where the temperature varies from −20° F. to 105° F. What allowance for this change of temperature must be made in the length of the walk if concrete changes .00000668 of its length for each Fahrenheit degree change of temperature, if the concrete is laid when the temperature is 50° F.?

11. Due to change in temperature a substance made of steel will change .0000065 of its length for each change of 1° F. What space should be left between the ends of steel rails 60 feet long when laid in a region where the temperature varies from −4° F. to 104° F., if the rails are laid when the temperature is 20° F.?

110. The Slide Rule. Introduction. The ***slide rule*** is an apparatus for determining mechanically products, quotients, powers, roots. It consists of two principal pieces so made that one may slide upon the other. It is simply a table of logarithms arranged so that they may be added or subtracted conveniently, and even more rapidly than with the ordinary printed table. Of course, the logarithms are not printed on the slide rule, but each number simply stands in the position indicated by its logarithm. This gives a

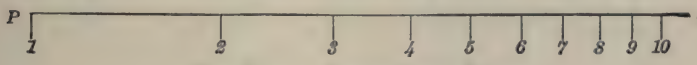


FIG. 91. — A LOGARITHMIC SCALE

scale which is not uniform and which is called a ***logarithmic scale***. Such a scale is shown in Fig. 91. This scale was graduated as follows:

The distance PQ was divided into 1000 equal parts. Since $\log 2 = .301$, 2 was placed at the 301st division, and as $\log 3 = .477$, 3 was placed at the 477th division, etc. for the succeeding integers.

To multiply numbers by the use of logarithms, we *add* the logarithms of the several factors and find the number corresponding to the sum of the logarithms. Hence, *to multiply two numbers by use of the slide rule*, it is only necessary

to place the logarithm of one of the numbers on one piece of the rule, end to end with the logarithm of the other number on the other piece of the rule, and then read off as the product the graduation mark at the end of these *combined* lengths.

For example, on the scale of Fig. 91 above, log 2, which is represented by the distance from 1 to 2, added to log 3, the distance from 1 to 3, gives log 6, the distance from 1 to 6. Thus, $2\times3=6$ on the scale.

Show in the same way that $2\times5=10$. Also that $(2.5)\times(3.5)=?$ Determine it first on the scale and compare with the result obtained by actual multiplication. Similarly $(2.5)\times(3.2)=?$

111. Description and History of the Slide Rule. One part of the rule is called the ***slide***. It slides along a groove in the center of the other part called the ***stock***. Some rules carry a ***runner*** for convenience in continued multiplication and division and squares and square roots. This is merely a sliding frame with a wire or line on it perpendicular to the length of the rule.

The device of having one piece *slide* on the other was invented by William Oughtred between 1620 and 1630, soon after the invention of logarithms by Napier in 1614.

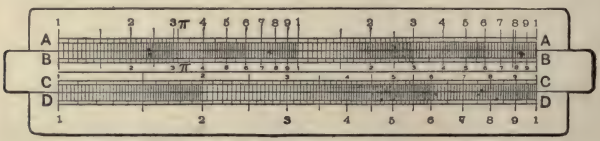


FIG. 92. — THE MANNHEIM ARRANGEMENT

Practically all slide rules in present use are ***Mannheim rules***. They are so named, not from any manufacturer, but from LIEUTENANT MANNHEIM of the French Army, who devised the present arrangement of the scales about 1850. Some higher priced rules have modifications of the Mannheim

arrangement, but the fundamental principles involved are the same as in the regular Mannheim rule. Fig. 92 shows the regular Mannheim arrangement.*

112. The Scales. Two scales are engraved along the upper edge of the groove. One, labeled A, is on the stock, the other, labeled B, is on the slide. These scales are identical. *The slide simply makes it possible to add graphically the segments on the scales.*

Along the lower edge of the groove are also two scales. One, C, is on the slide, the other, D, is on the stock. These two scales are likewise identical.

Scales A and B range from 1 to 100 while C and D range only from 1 to 10. The range on one set being smaller, the number of divisions can be increased. Thus C and D give greater accuracy in reading them since there are more divisions. Notice also that every number on the A and B scales is the *square* of every number below it on the C and D scales, due to the fact that the unit on one is twice as great as on the other. This gives us at once *a table of squares and square roots.* The *runner* enables one to find corresponding numbers on the upper and lower scales, and also to hold a given position during a repeated operation.

113. Operations with the Slide Rule. To use a slide rule successfully, one must acquire the virtues of speed and accuracy. This requires practice, since the accuracy of a result often requires the estimation of a number falling between the lines of division.

* Fig. 92 is reproduced on a larger scale on the first fly leaf at the back of the book. By cutting out this leaf and carefully cutting up the figure, a slide rule can be made by the student. It will not be very accurate, but it will suffice to illustrate the principles.

Slide rules, varying from the inexpensive pasteboard kind to the higher priced wood and celluloid ones used by engineers, may be had at practically all stores which carry draftsmen's supplies.

A skillful operator, with a 10-inch slide rule, can always secure results accurate to three significant figures. This is accurate enough for most of the purposes of applied science.

The beginner should use small numbers until he becomes familiar with the operations.

114. Multiplication. Multiply 2 by 3. Move the slide so as to put 1 of the *B* scale on 2 of the *A* scale; then above 3 on the *B* scale read the product 6 on the *A* scale. This is simply adding logarithms.

Notice that *the 1 at either end of the B scale may be used.* Try both ways. Use that 1 which brings the second factor under scale *A*.

For example, multiply 65 by 3 using *CD* scales. If we set the left end 1 of *C* on 65 *D*, then 3 *C* is off the scale *D*. In this case put the *right* end 1 of *C* on 65 *D* which brings 3 *C* under 195.

The decimal point is placed by inspection, *i.e.* by making an approximate mental computation to determine the number of integers or the number of decimal places.

For example, multiply 18.5 by 2.8. Set 1 *C* on 185 *D* and under 28 *C* read 518 *D*. But the product is roughly $18 \times 3 = 54$; therefore we know that there are only two places before the decimal point in the product of 18.5 and 2.8 Thus 518 read on *D* means 51.8

115. Division. Divide 6 by 2. Set 2 *C* on 6 *D*, under 1 *C* read the quotient 3 on *D*. This means merely subtracting logarithms.

Also, divide 7.2 by 2.5. Set 25 *C* on 72 *D*, under 1 *C* read the quotient 2.9 on *D*. Therefore,

To divide one number by another, set the divisor on scale C on the dividend on scale D, and under 1 *C read the quotient on D.*

116. Combined Multiplications and Divisions. Find the value of

$$\frac{(25.2)(9.6)}{8.4}.$$

Set 84 C on 252 D, and under 9.6 C read the quotient 28.8 on D.

Notice that 84 C on 252 D gives the quotient under 1 C but as this is to be multiplied by 96, 1 C is already on this quotient and hence we have only to read the product on scale D under 9.6 C.

An important application of this is *finding the fourth term of a proportion.* For example, from the proportion

$$8.4 : 25.2 = 9.6 : x,$$

we find

$$x = \frac{(25.2)(9.6)}{8.4}.$$

The rule then to find the fourth term is obvious, viz.:

To find the fourth term of a proportion, set the first term on the second, and under the third term read the fourth.

In continued multiplications and divisions, the runner with its perpendicular indicator is quite convenient.

EXAMPLE 1. Find the value of $2 \times 3 \times 4$.

Set 1 C on 2 D, set runner on 3 C (marking this product). Now set 1 C at right end on runner and under 4 C read 24 on D. Thus $2 \times 3 \times 4 = 24$.

EXAMPLE 2. Find the value of $\frac{56}{7 \times 4}$.

Set 7 C on 56 D, set runner on 1 C (marking the quotient), set 4 C on the runner and under 1 C read the result 2 on D.

EXAMPLE 3. Find the value of $\frac{24 \times 51}{17 \times 12}$.

Set 17 C on 24 D, set runner on 51 C, set 12 C on runner, under 1 C read result 6 on D.

117. Squares and Square Roots. Scales C and D were constructed on a unit twice as large as that for scales A and B. Accordingly, every number on the A and B scales is the square of the number vertically below it on the C and D scales. Thus above 2 C we find 4 A, above 3 is 9, above 25 is 625, etc. Thus,

To square any number, find the number on scale C and read its square vertically above it on scale B.

To extract the square root of a number, find the number on scale B and read its square root vertically below it on scale C.

EXAMPLE 1. Find the value of $16\sqrt{3}$.

Set 1 C on 3 A, under 16 C read the result 27.7 on D.

EXAMPLE 2. Find the value of $16/\sqrt{5}$.

$$\frac{16}{\sqrt{5}} = \frac{16}{5}\sqrt{5}.$$

Set 5 C on 5 A, under 16 C read result 7.16 on D.

EXAMPLE 3. Find the area of a circle whose radius is 3 ft. Set 1 C on 3 D, above π on B read the area 28.3 sq. ft., on A.

EXERCISES

Find the value of each of the following expressions.

1. $(4.67)(8.32)$.
2. $(.341)(3.86)$.
3. $\dfrac{31.5}{16.2}$.
4. $\dfrac{(97)(15)}{23}$.
5. $\dfrac{849}{31.6}$.
6. $\dfrac{(18.6)(3.4)}{3.1416}$.
7. $\dfrac{27\sqrt{19}}{37}$.
8. $(47.2)(18.1)(.47)$.

9. Compute W, the safe load in tons, when uniformly distributed on white oak beams, from the formula

$$W = \frac{2}{3}\frac{bd^2}{l},$$

if b (the breadth in inches) is 3, d (the depth in inches) is 8, and l (the distance in inches between supports) is 192.

10. Find the area of a circle whose radius is 75 millimeters.

CHAPTER VIII

THE PROGRESSIONS

118. Introduction. Progressions are among the oldest topics of all mathematics. Egyptian scholars had studied them thousands of years before the Christian era. By 2000 B.C. they had attained to a considerable knowledge of them, as evidenced in the papyrus of Ahmes.

Progressions are important, not only in the solution of numerical problems, but also in theoretical developments. For example, out of the association of arithmetic and geometric progressions grew the subject of logarithms.

119. Arithmetic Progression. An ***arithmetic progression*** (written $A.P.$) is a succession of terms so related that any one, after the first, is obtained from the preceding one by adding a constant number. This constant is known as the ***common difference.*** Thus

2, 5, 8, 11, $\cdots$, with a common difference 3,

5, 1, -3, -7, $\cdots$, with a common difference -4,

a, $a+x$, $a+2x$, $\cdots$, with a common difference x,

are arithmetic progressions.

Is 2, 3, 5, 4, 7, 11, $\cdots$, an arithmetic progression?

120. Terms of an Arithmetic Progression. In such a succession of terms as just described, we are interested in

the ***first term***, known as a,
the ***common difference***, known as d,
the ***last or nth term***, known as l,
the ***number of terms***, known as n,
and the ***sum of the terms***, known as s.

Therefore, we may write the arithmetic progression in general as follows.

1st term	*2d term*	*3d term*	*17th term*	*nth or last term*
a,	$a+d$,	$a+2\,d$, $\cdots$,	$a+16\,d$, $\cdots$,	$a+(n-1)d$.

It thus appears that *any term is equal to the first term, plus the common difference multiplied by the number of the term less one*, i.e.

$$l=a+(n-1)d.$$

For example, the 17th term $=a+(17-1)d=a+16\,d$.

EXERCISES

1. Given the A. P. 3, 7, 11, $\cdots$. What is the common difference? What is the 10th term?

Begin with the 10th or last term and write the series backwards.

2. Do as in Ex. 1 for the A. P. 1, 4, 7, $\cdots$ to 10 terms.

3. Do as in Ex. 1 for the A. P. a, $a+x$, $a+2\,x$, $\cdots$ to 10 terms.

4. If l is the last term and d the common difference, write the A. P. backwards.

[HINT. If l is the last term, then $l-d$ is the next to the last, etc.]

5. Write the A. P. which exhibits the strokes of a clock striking the hours.

121. Sum of the Terms of an Arithmetic Progression. The sum of an A. P., *i.e. the sum of all the terms taken*, is found as follows.

$$s=a+(a+d)+(a+2\,d)+\ \cdots \text{ to } n \text{ terms},$$

or, written backwards,

$$s=l+(l-d)+(l-2\,d)+\ \cdots \text{ to } n \text{ terms}.$$

Addition of corresponding terms gives

$$2\,s=(a+l)+(a+l)+\ \cdots \text{ taken } n \text{ times}=n(a+l).$$

Thus, $$s=\frac{n}{2}(a+l).$$

EXAMPLE 1. Find the total number of strokes made by a clock in striking 12 hours.

Here $a=1$, $l=12$, $n=12$; therefore $s=\frac{1}{2}^2(1+12)=78$.

EXAMPLE 2. What is the total price paid for a lot bought on the installment plan of 25¢ the first week, 50¢ the second week, 75¢ the third week, etc.; to continue for one year of 52 weeks?

Here $a=25$, $d=25$, $n=52$; hence the last term is

$$l=25+(51)25=1300¢,$$

$$s=\frac{n}{2}(a+l)=26(25+1300)=34{,}450¢=\$344.50.$$

EXERCISES

Find the last term and the sum of all of the terms in each of the following A. P.'s.

1. 5, 9, 13, ··· to 20 terms. *Ans.* $l=81$, $s=860$.

2. 1, 3, 5, 7, ··· to 25 terms.

3. $2\frac{1}{3}$, $3\frac{2}{3}$, 5, ··· to 15 terms.

4. $\frac{1}{4}$, $-\frac{3}{4}$, $-\frac{7}{4}$, ··· to 16 terms.

5. Find the sum of the first 50 even integers beginning with 0.

6. A row of twenty trees is set at intervals of fifteen feet, beginning at a point forty feet from a house. How far is the last one from the house? If a man carries a bucket of water from a well by the house to each tree, how far does he walk?

7. Write an A. P. of nine terms for which the first term is 4 and the last term 20.

[HINT. Use $l=a+(n-1)d$ where $a=4$, $l=20$, etc.]

8. How far apart must 9 plants be set to be equally spaced in a row 16 ft. long? *Ans.* 2 ft.

9. If there are sixteen plants in a row 40 feet long, how far apart are they?

10. If the sum of an A. P. is $18\frac{3}{4}$, the first term $\frac{3}{4}$, the number of terms 10, find l and d.

11. A man deposits \$75 in a savings bank. How much must this and later deposits be increased each year so that in 10 years his total deposit shall amount to \$1875, not counting interest?

12. A farmer has \$100 set aside for drilling a well. How deep could he have it drilled at 25¢ for the first foot and for each additional foot 1¢ more than the preceding one?

13. A freely falling body descends 16 ft. the first second and approximately 32 ft. more each second thereafter than in the preceding second. How deep is a well to the bottom of which a stone falls in 3 sec.? In 6 sec.? How far does it fall in the *sixth second?*

14. A bomb is released from an airplane 1 mile high. How long does it take the bomb to reach the ground? How far did it go the last second?

If the airplane is going 75 miles per hour, how far horizontally in advance of its starting point does the bomb hit the ground?

15. According to the law of falling bodies as given in Ex. 13, what is the relation between distance, s, and time, t, *i.e.* between the sum of the A. P. and the number of terms?

122. Geometric Progression. A ***geometric progression*** (written G. P.) is a succession of terms so related that any term, after the first, is obtained by *multiplying* the preceding term by a constant number called the ***ratio***. Thus

$2, 8, 32, \cdots$, with ratio 4,

and $-2, 6, -18, 54, \cdots$, with ratio -3,

are geometric progressions.

What is the ratio for each of the following geometric progressions:

$9, 3, 1, \frac{1}{3}, \cdots$?

$4, -12, 36, -108, \cdots$?

$2, \frac{4a}{b}, \frac{8a^2}{b^2}, \cdots$?

123. Terms of a Geometric Progression. In a geometric progression, we are interested in

the ***first term***, known as a,
the ***ratio***, known as r,
the ***last or nth term***, known as l,
the ***number of terms***, known as n,
the ***sum of the terms***, known as s.

Therefore, we may write, generally, the series of terms of a G. P. as follows.

1st term	*2d term*	*3d term*	*17th term*	*nth or last term*
$a,$	$ar,$	$ar^2,$	$\cdots\ ar^{16},$	$\cdots\ ar^{n-1}.$

The exponent of r in any term is one less than the *number* of that term. The *nth or last term* is

$$l = ar^{n-1}.$$

For example, the 17th term is ar^{16}.

EXERCISES

Which of the following sequences of numbers represent geometric progressions? Find the ratio and the last term in each such progression.

1. $\frac{1}{4}, \frac{1}{2}, 1, \cdots$ to 9 terms.
2. 2, 3, 5, 4, 7, $\cdots$ to 6 terms.
3. $2, \frac{2}{5}, \frac{4}{25}, \cdots$ to 6 terms.
4. 7, .7, .77, $\cdots$ to 5 terms.
5. 72, .72, .0072, $\cdots$ to 5 terms.
6. $(a+b), (a+b)^2, (a+b)^3, \cdots$ to 6 terms.
7. $(a+b), (a^2-b^2), (a^3-ab^2-a^2b+b^3), \cdots$ to 4 terms.
8. $\sqrt{2}, 2, \sqrt{8}, \cdots$ to 7 terms.
9. If a and ar^2 are the end terms, respectively, of a G. P. of *three* terms, what is the middle term? What is the middle term if 4 and 9 are the end terms of a G. P. of three terms?
10. If 4 is the *first* term of a G. P. and $\frac{2}{3}\frac{7}{2}$ is the *fourth* term, find the second and third terms.

[HINT. Let $a=4$, and $ar^3=\frac{2}{3}\frac{7}{2}$; find r.]

124. Sum of the Terms of a Geometric Progression. By definition the sum of n terms of a G. P. is

$$s = a+ar+ar^2+\ \cdots\ +ar^{n-1}.$$

Therefore $$rs = ar+ar^2+ar^3+\ \cdots\ +ar^n.$$

Subtracting these gives

$$s-rs = a-ar^n = a-rar^{n-1} = a-rl.$$

Thus $$s = \frac{a-ar^n}{1-r} = \frac{a-rl}{1-r}.$$

EXAMPLE. In the G. P. 6, 2, $\frac{2}{3}$, $\cdots$ find the 6th term and the sum of 6 terms.

Here $a=6$, $r=\frac{1}{3}$, $n=6$, $l=ar^{n-1}=6(\frac{1}{3})^5=6\frac{1}{243}=\frac{2}{81}$.

Hence s is given either by the formula

$$s=\frac{a-ar^n}{1-r}=\frac{6-6(\frac{1}{3})^6}{1-\frac{1}{3}}=\frac{728}{81},$$

or by the formula

$$S=\frac{a-rl}{1-r}=\frac{6-\frac{1}{3}\cdot\frac{2}{81}}{1-\frac{1}{3}}=\frac{728}{81}.$$

EXERCISES

1. If $a=3$, $r=\frac{1}{2}$, $n=8$, find l and s.

2. If $a=6$, $n=6$, $l=-\frac{64}{81}$, find r and s.

3. How many terms of the G. P.

$$1¢+2¢+4¢+\cdots$$

are necessary so that the sum shall be at least \$40?

Ans. 12 terms.

4. A blacksmith agrees to shoe a horse as follows: $\frac{1}{4}¢$ for the first nail driven, $\frac{1}{2}¢$ for the next, $1¢$ for the next, etc., for the 8 nails of one shoe, and a like amount for each of the other three shoes. What does he receive for the work? *Ans.* \$2.55

5. If the first term of a G. P. of 5 terms is 2 and the last term 1250, find the ratio and the sum of 5 terms.

6. Find the sum of 7 terms of $1+2^{-1}+2^{-2}+\cdots$.

7. If $a=2$, $n=5$, $l=1250$, find r and s.

8. If $a=17$, $l=459$, $n=4$, find r and s and write the series.

9. The common housefly will lay, incubate, and mature a litter of eggs about every $3\frac{1}{2}$ weeks. What will be the number of descendants of one female fly in 14 weeks if there are 150 to a litter, evenly distributed as to sex?

10. Find the amount of \$1 at 4% compound interest for 1 year. For 2 years. For 3 years. For 4 years.

[HINT. The rate is $4\%=.04$ and the amount is \$1.04 for one year. This is the principal for the second year. The *amount*, therefore, for the second year is (1.04) times $(1.04)=(1.04)^2$.]

Do these results form a geometric progression?

11. Find the amount of p dollars at rate r compound interest for 1 year. For 2 years. For 3 years. For n years.

Ans. The amount for n years $= p(1+r)^n$.

[HINT. The amount of \$1 for 1 year at rate r is $\$1(1+r)$ and the amount of $\$p$ is therefore p times this, *i.e.* $\$p(1+r)$. The amount for the second year is therefore $(1+r)$ times the new principal $\$p(1+r)$, *i.e.* it is $p(1+r)^2$.]

12. Find the amount of \$50 for 8 years at 5% compound interest.

[HINT. The computation is shortened by use of logarithms.]

Ans. $\$50(1.05)^8$.

13. A man deposits \$50 *each year* in a building and loan association at 5% compound interest. What is the total amount of interest and deposits at the end of 6 yr.?

[HINT. The first deposit is compounded at 5% for 6 yr. and thus amounts to $50(1.05)^6$. The second deposit amounts to $50(1.05)^5$, being on interest for 5 yr. The next deposit amounts to $50(1.05)^4$, etc. The total amount for the six years is thus

$$50(1.05)+50(1.05)^2+\cdots+50(1.05)^6.$$

But this is a G. P. of six terms. Show that its sum is

$$50(1.05)\left[\frac{(1.05)^6-1}{.05}\right].$$

Compute this amount by use of logarithms.]

14. Find the total amount of a yearly deposit of p dollars compounded at rate r for n years.

Ans. $p(1+r)\left[\frac{(1+r)^n-1}{r}\right]$.

15. On a certain life insurance endowment policy for \$1000, the insured agrees to pay \$47.68 each year for 20 years, and he is to receive \$1000 at the end of the period. What is the total amount of these deposits compounded annually at $3\frac{1}{2}\%$? What does the company obtain at the end of the period for carrying the risk, if it earns just $3\frac{1}{2}\%$ on money in its possession? *Ans.* Amt. = \$1395.44

16. A man buys a machine for \$150. What amount of money should be deposited each year to draw 4% compound interest so that the amount of these deposits would purchase a similar new machine at the end of 12 years? *Ans.* \$9.60

[HINT. Use the formula of Ex. 14, solving for p.]

17. What equal annual deposit at 4% compound interest will amount to \$300 in 15 years?

18. What amount of money could profitably be expended for a harvesting machine which will last 15 years, if it saves annual labor bills of \$40, money being worth 5%.

[HINT. The total saving, with interest from the *end* of each year, is given by the sum of the geometric progression

$$s=40(1.05)^{14}+40(1.05)^{13}+\cdots+40=40[(1.05)^{15}-1]\div(.05).$$

But this is the saving at the *end* of 15 years. The principal p invested now would produce in 15 years the amount $p(1.05)^{15}$. It follows that $p(1.05)^{15}=s$. Solve for p, using logarithms.] *Ans.* $p=\$415.20$

125. Infinite Geometric Progressions. Thus far we have considered only those progressions which have a *limited* number of terms. There are many problems and phenomena whose discussion calls for a series of an *unlimited* number of terms. For example, suppose an elastic ball falls 4 feet and rebounds half of that distance, then falls 2 feet and rebounds half of that distance, etc., rebounding each time half of the distance fallen. The entire distance fallen would be the sum of the series

$$4+2+1+\tfrac{1}{2}+\tfrac{1}{4}+\cdots \text{ to infinity.}$$

This is a G. P. in which the ratio is less than unity and the number of terms is unlimited.

If, in a G. P., $r<1$, each term is smaller than the preceding, the last term l approaches zero as the number of terms increases, and the formula $s=\dfrac{a-rl}{1-r}$ becomes, when $l=0$,

$$S=\textit{limit of } s=\frac{a-r\cdot o}{1-r}=\frac{a}{1-r}.$$

EXAMPLE. Find the sum of the infinite series

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\cdots.$$

Here $a=1$, $r=\frac{1}{2}$ and $S=\dfrac{a}{1-r}=\dfrac{1}{1-\frac{1}{2}}=2.$

This simply means that as the number of terms of this series increases, the *sum* of them differs less and less from 2.

EXERCISES

1. Find the sum of the following infinite geometric progressions.

(*a*) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots$.

(*b*) $1+\frac{3}{4}+\frac{9}{16}+\cdots$.

(*c*) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots$.

(*d*) $\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\cdots$.

The progression (*d*) is (*c*) transformed by subtracting successive pairs of terms.

2. What common fraction gives rise to the repeating decimal $.44\cdots$?

[Hint. Find the sum of the series

$$.4+.04+.004+\cdots.]$$

3. What common fraction gives rise to $.777\cdots$? To $.2727\cdots$? To $.315315\cdots$?

4. A farm returns a net income of \$800 per year. Considering that money is worth 5% and that this return can be kept up indefinitely, what is the farm worth? How much is it worth if money is worth 4%? 6%?

[Hint. The amount of \$1.00 for 1 year at 5% is \$1.05. Therefore the present value of an amount due in 1 year is that amount divided by 1.05. The amount of \$1.00 for two years at 5% is $\$(1.05)^2$, and the present value of \$1 two years hence is $1/(1.05)^2$. The present value of the first return of \$800 due in 1 year is therefore $\$800/(1.05)$; the present value of the second return due in two years is $\$800/(1.05)^2$; etc., indefinitely. The sum of the series $800/(1.05)+800/(1.05)^2+\cdots$ is

$$\frac{800}{1.05}\div\left(1-\frac{1}{1.05}\right)=\frac{800}{1.05}\div\frac{.05}{1.05}=\frac{800}{1.05}\times\frac{1.05}{.05}=\frac{800}{.05}=\frac{800}{5}\cdot 100.$$

This gives the following rule for determining the value of property that yields a fixed income: *Multiply the income by* 100 *and divide by the number of per cent in the assumed rate of interest.*]

5. Find the value of a farm that yields a net income of \$2400, with money at 5%. At 8%. Does a higher interest rate give lower or higher valuations?

6. Suppose the farm in Ex. 4 produces \$800 net return only on alternate years, *i.e.* it lies fallow every other year. What is its value with money at 5%?

7. A ball falls 6 ft. and at each fall rebounds 2/5 of the distance. Find the distance the ball has traveled before coming to rest.

8. Find the time it takes the ball in the previous exercise to come to rest using $s = 16\,t^2$ as the law of falling bodies where s = distance fallen in t seconds.

9. The mid-points of the sides of a square 12″ on a side are joined to form a smaller square. Then the mid-points of the sides of this smaller square are joined to form a still smaller square. This process is continued indefinitely. What is the sum of the perimeters of these squares?

CHAPTER IX

COMPOUND INTEREST—ANNUITIES—DEPRECIATION

126. Compound Interest. When the interest on a note or an account is added to the principal as it becomes due, and the new amount draws interest, the owner is said to receive *compound interest.* Interest is computed in this way by building and loan associations, life insurance companies, large corporations, etc., since they can reinvest at once any deposits, receipts, premiums, etc.

Compound interest is no longer allowed on notes. But if any one collects his interest when due, and at once reinvests it, he virtually has the advantage of compound interest.

In computing interest semiannually, the rule is to take twice the time and one half the rate, since the payments are made twice as often, and are only half as large. For computing interest payable quarterly, take four times the time and one fourth the rate. Proceed in a similar manner for any specified number of payments per year.

127. Amount of $p at Rate r Compounded for n Years. The amount of \$1 at rate r for 1 year is $\$(1+r)$. Then the amount of $\$(1+r)$ for the next year is $\$(1+r)(1+r)=\$(1+r)^2$. The amount of $\$(1+r)^2$ for the next year is $\$(1+r)^2(1+r)=\$(1+r)^3$. This is the amount of \$1 for 3 years. For 4 years it is $\$(1+r)^4$, etc. For n years the amount is $\$(1+r)^n$. Then the amount of $\$p$ for n years at rate r is

$$A=p(1+r)^n. \tag{1}$$

EXAMPLE. Find the amount of \$500 deposited in a savings bank at 5% annually, the interest compounded semiannually for 2 years.

The semiannual rate is $2\frac{1}{2}\%$, and is computed for four periods.

FIRST SOLUTION: As in Exs. 10, 11, § 124, the amount of \$500 for 4 periods @ $2\frac{1}{2}\% = \$500(1.025)^4$. This may be computed by ordinary arithmetic; but the work is shortened by the use of logarithms.

$$\log 500 = 2.6990$$
$$4 \log (1.025) = \quad .0428$$
$$\text{Sum} = 2.7418 = \log (551.90).$$

Thus the compound amount is \$551.90

NOTE. In operations involving numbers of five or six figures, a five-place or a seven-place table of logarithms should be used. If the time interval is over twenty years, at least a five-place table is desirable.

SECOND SOLUTION: By a compound interest table (see Appendix), the amount of \$1.00 for 4 periods @ $2\frac{1}{2}\% = \$1.10381$ and the amount of \$500 for 4 periods @ $2\frac{1}{2}\% = \$500(\$1.10381) = \$551.90$

EXERCISES

1. Find the amount of \$150 at 5% interest compounded annually for 3 years.

2. Use logarithms to find the amount of \$50 at 6% compound interest for 5 years. *Ans.* \$66.91

[HINT. The rate is 6%, hence $r = .06$, and $1 + r$ is 1.06]

3. Solve Ex. 2 if the interest is compounded semiannually.

[HINT. Use 10 periods at 3%.]

4. A father deposits \$25 in a building and loan association for his son at birth. It draws 5% compound interest payable semiannually. What is the amount due the son when he is 21 years old?

5. When a son is 10 years old, his father agrees to deposit for him in a building and loan association, beginning at once and on each succeeding birthday, a sum of money equal to his age. The deposit is to bear 4% compound interest payable semiannually. What is the amount to his credit when he is 21 years old? *Ans.* \$226.32

[HINT. The first deposit of \$10 bears interest for 11 years, etc. The son is also to be credited with \$21 on his twenty-first birthday.]

6. A man deposits \$50 at the beginning of each year in a building and loan association which pays 5% per year, payable semiannually. If the interest and the several principals are allowed to accumulate, what will be the total amount at the end of 10 years?

128. Annuities. An ***annuity*** is a series of *equal payments* made at *equal intervals of time* during a *fixed term of years.*

The first payment is supposed to be made at the *end* of the first year unless otherwise specified.

Annuities are usually computed at compound interest and are therefore applications of geometric series (§ 124). The whole subject of annuities is of great fundamental importance, since its practical applications include farm loans, rents, leases, pensions, dowers, life-estates, salaries, life insurance, old age insurance, disability insurance, and many other similar financial operations.

129. Farm Loans. Under the recent Farm Loan Act instituted by the Federal Government, a farmer may borrow from the Government on long time and at a nominal rate of interest. A problem which then arises with him is to pay off this loan, both principal and interest, by a series of equal payments extending over a period of years specified in the agreement.

For example, a farmer buys a farm for $10,000 and pays for it by a loan from the Government. He may wish to know *now* how large a payment to make each year, say for 30 years, so that at the end of that time the loan will be cancelled. The problem may be solved by considering how large an annuity $10,000 will purchase at 5% interest, for example, for the 30 years. The answer to this particular problem is $650.50 per year.

Notice also that this $650.50 per year pays the interest on the loan each year, $500, and besides pays $150.50 on the principal. This payment of $150.50 each year at 5% for 30 years must therefore accumulate at compound interest to precisely $10,000 at the end of the period. This it does and the loan is accordingly cancelled.

130. Problem A. Amount of an Annuity. *To find the amount of $1 payable at the end of each year for n years.*

The first payment made at the end of the first year draws interest at rate r, for $n-1$ years, and amounts, by § 127, to $(1+r)^{n-1}$.

The second payment, similarly, for $n-2$ years amounts to $(1+r)^{n-2}$.

. . . . . .

The next to last payment, for 1 year, amounts to $(1+r)^1$.

The last payment at the end of the term is 1.

The sum of these amounts is the amount desired. It is usually called $s_{\overline{n}|}$. Thus

$$s_{\overline{n}|}=1+(1+r)+(1+r)^2+\cdots(1+r)^{n-1}.$$

By § 124, the sum of this series of terms is

$$(2) \qquad s_{\overline{n}|}=\frac{(1+r)^n-1}{r}.$$

If the payment each time is p, the total amount is

$$(3) \qquad ps_{\overline{n}|}=p\left(\frac{(1+r)^n-1}{r}\right).$$

If payments are made at the BEGINNING *of each year,* as is often the case, the amount at the end of n years is denoted by ${}_{\overline{n}|}s$.

The amount is the same as if the preceding amount $s_{\overline{n}|}$ were allowed to run another year; hence

$$(4) \qquad {}_{\overline{n}|}s=(1+r)s_{\overline{n}|}=(1+r)\left(\frac{(1+r)^n-1}{r}\right).$$

Formulas (2), (3), and (4) can be computed from a *compound interest table* (see Appendix), since such a table gives the value of $(1+r)^n$, the amount of 1 compounded annually for n years at rate r.

EXAMPLE 1. Find the accumulated value of an annuity of \$500 a year at 5% compounded annually for 15 years, payments being made at the end of each year.

FIRST SOLUTION: From equation (3), the desired amount is

$$s_{\overline{n}|} = \$500\left[\frac{(1+r)^n - 1}{r}\right] = \$500\left[\frac{(1.05)^{15} - 1}{.05}\right].$$

From a *four-place* table of logarithms,

$$15 \log (1.05) = .3180 = \log (2.0795),$$

that is $$(1.05)^{15} = 2.0795$$

Thus, $$s_{\overline{n}|} = \$500\,\frac{2.0795 - 1}{.05} = \$10{,}795.*$$

SECOND SOLUTION: By a compound interest table, the value of

$$(1.05)^{15} = 2.07893,$$

and $$s_{\overline{n}|} = \$10{,}789.30$$

EXAMPLE 2. Calculate the accumulation in the previous example if the payments were made at the *beginning* of each year.

From equation (3), $_{\lceil\overline{n}|}s$, the amount desired in this case $= (1+r)s_{\overline{n}|}$. Therefore,

$$_{\lceil\overline{n}|}s = (1.05)(\$10{,}789.30) = \$11{,}328.76$$

EXERCISES

1. Find the accumulation of \$150.50 per year at 5% for 30 years, payments being made at the end of the year. *Ans.* \$9999.05

2. A man deposits at the *beginning* of each year \$150.50 in a building and loan association at 5% compound interest. What is the total value at the end of 30 years? *Ans.* \$10,499.

3. What will a farmer save in 30 years, if his yearly saving is \$300 and is compounded at 5%? *Ans.* \$19,931.65

4. Find the amount of an annuity of \$100 at the end of each year for 10 yr. at 5% interest, compounded semiannually.

Ans. \$1261.47

5. Find the amount of \$50 at the end of every six months for 10 years at 5% interest, compounded semiannually. *Ans.* \$1277.23

* By use of five-place table of logarithms, $s_{\overline{n}|}$ in this example is equal to \$10,789.30

131. Problem B. Present Worth of an Annuity. *To find the present value of an annuity of 1 per annum for n years.*

The present value, x, of 1 due in n years is the sum which will amount to 1 in n years with compound interest at rate r. Therefore

$$x(1+r)^n=1, \quad i.e. \quad x=\frac{1}{(1+r)^n}.$$

Thus

the present value of the first payment, receivable in 1 year, is $1/(1+r)$;

the present value of the second payment, receivable in 2 years, is $1/(1+r)^2$;

the present value of the nth payment, receivable in n years, is $1/(1+r)^n$.

But the present value of the whole annuity is the sum of the present values of the several payments. This sum may be denoted by $a_{\overline{n}|}$. Thus

$$(5) \qquad a_{\overline{n}|}=\frac{1}{1+r}+\frac{1}{(1+r)^2}+\cdots\frac{1}{(1+r)^n}.$$

This is a geometric series whose first term is $1/(1+r)$ and whose ratio is $1/(1+r)$. Hence, by § 124, we find after simplifying,

$$(6) \qquad a_{\overline{n}|}=\frac{1-(1+r)^{-n}}{r}=\frac{(1+r)^n-1}{r}\cdot\frac{1}{(1+r)^n}=\frac{s_{\overline{n}|}}{(1+r)^n}.$$

If each payment is p instead of 1, the present value of such an annuity is merely p times $a_{\overline{n}|}$ in equation (6).

Equation (6), in its final form, shows that the present value of an annuity of \$1 is merely the amount of an annuity for \$1 for n years divided by the amount of \$1 at compound interest for n years. In this form it may be computed either by arithmetic, or, far more easily, from a compound interest table.

132. Problem C. Present Price of an Annuity. *To find that annual payment for n years which $1 will purchase.* Let p = the annual payment continued for n years, at rate r. Its present or cash value is its cost, *i.e.* \$1 in this case. Therefore, by equation (6), § 131,

$$\$1 = p\left[\frac{(1+r)^n-1}{r} \cdot \frac{1}{(1+r)^n}\right].$$

Hence the annual payment, p, for n years, which \$1 will purchase is

$$p = \frac{r \cdot (1+r)^n}{(1+r)^n-1},$$

and the annual payment, P, which $\$A$ will purchase is

$$P = \frac{A \cdot r(1+r)^n}{(1+r)^n-1}. \tag{7}$$

EXAMPLE 1. What annual payment for 10 years will \$500 purchase, if money is worth 5%?

SOLUTION: $P = \dfrac{500 \cdot (.05)(1.05)^{10}}{(1.05)^{10}-1} = \dfrac{(500)(.05)(1.62890)}{(1.62890)-1}$,

from a compound interest table. Therefore $P = \$64.75$

EXERCISES

1. What is the present value of \$600 due in 8 years if money is worth 4% compounded annually? *Ans.* \$438.41

2. Solve the previous example if the interest is computed at 4% compounded semiannually. *Ans.* \$437.07

3. Find the present value of an annuity of \$53 receivable at the end of each year for 20 years, at 5%, (*a*) by logarithms and (6), § 131, (*b*) by a compound interest table. *Ans.* \$660.50

4. A man buys a farm and agrees to pay for it \$1000 in cash and \$500 at the end of each year for 10 years. If money is worth 6%, what would be the equivalent cash price of the farm? *Ans.* \$4680.44

5. What annual payments continued for 10 years are equal to \$1000 cash, if money is worth 5%? *Ans.* \$129.50

[HINT. Let p = the annual payment. Then from (6), since $a_{\overline{n}|} = \$1000$,

$$\$1000 = p\left[\frac{(1.05)^{10}-1}{(.05)} \cdot \frac{1}{(1.05)^{10}}\right],$$

from which p may be computed.]

6. What annual payments for 20 years can be purchased for \$500, money being valued at 5%? *Ans.* \$40.12

7. What semiannual payments continued for 20 years can be purchased for \$500, at 5%? *Ans.* \$19.92

8. A man buys a farm for \$10,000 and wishes to pay for it by a series of equal annual payments of such amount as to pay both principal and interest in 30 years at 5%. Find the amount of the annual payment. *Ans.* \$650.50

[HINT. Find what annual payment \$10,000 will purchase.]

9. A man secures \$2400 for the purchase of farm equipment by a loan from the government. He agrees to pay \$20 per month, including 6% interest on all sums remaining unpaid. How long will it take to pay off the loan and what will be the amount of the last payment?

Ans. 184 months; last payment, \$14.26

10. A farmer wishes to provide an income for old age. He estimates that at age 30 years he will have 35 years of productive activity ahead of him, and that he can save \$250 per year during that time. This accumulation at 5% compound interest at age 65, together perhaps with \$2500 cash available from the sale of his farm and tools, will purchase what annual payments for 15 years, if money is worth 5%?

Ans. \$2416.27

11. Solve the previous exercise, estimating that the farmer can save \$100 per year for 30 years at 5% and that he will use this accumulation at age 60 years, plus \$2500, to purchase an income for 20 years at 5%. What annual payments can he thus purchase? *Ans.* \$733.73

133. Problem D. Sinking Fund. *To find the annual payment that will amount to \$A in n years.* Such annual payments made to accumulate a given amount at a given time constitute a ***sinking fund.***

Let p denote the annual payment, and A the amount at the end of n years. Then, by (3), § 130, the amount is

$$p\frac{(1+r)^n-1}{r}=A, \tag{8}$$

whence

$$p=\frac{Ar}{(1+r)^n-1}. \tag{9}$$

EXERCISES

1. A farmer wishes to provide \$1500 for the replacement of machinery ten years hence. How much must he set aside each year at 5% compound interest to meet this contingency? *Ans.* \$119.26

2. How much would the farmer of Ex. 1 need to set aside every six months to provide for the same amount, if the interest is compounded semiannually? *Ans.* \$58.72

3. A township issues bonds for \$100,000, payable at the end of 25 years, for road improvements. What sum must be set aside each six months to meet the payment of these bonds when due, if the sums set aside bear 4% interest? *Ans.* \$1182.30

4. A man owes a note of \$640 due at the end of 5 years with simple interest at 6%. He proposes to meet the payment of both principal and interest at the end of the time by depositing a certain amount every six months in a savings association, at 5% compound interest. Compute the amount to be thus deposited. *Ans.* \$74.26

134. Perpetuities. Capitalization. When the payments of an annuity are continued indefinitely, it is called a ***perpetuity.*** Perpetuities are important in practical business affairs. An annual income from the rent of property, from an endowment fund, or from a fixed salary, has a definite cash equivalent. Likewise, an annual outgo continued indefinitely has a cash equivalent. The process of finding such cash equivalents is called ***capitalization.***

Problem E. *To find the present value of a perpetuity whose payments are made annually.*

Let p = the annual payment due 1 yr. hence, and thereafter annually indefinitely.

The present value of the first payment due in 1 year is $p/(1+r)$;
The present value of the second payment due in 2 years is $p/(1+r)^2$;
The present value of the third payment due in 3 years is $p/(1+r)^3$;

and so forth indefinitely. The present value of the perpetuity is the sum of these present values of the separate payments. If s_∞ denote this sum, we have

$$s_\infty = \frac{p}{1+r} + \frac{p}{(1+r)^2} + \frac{p}{(1+r)^3} + \cdots. \tag{10}$$

This is an infinite geometric series whose first term is $p/(1+r)$, and whose ratio is $1/(1+r)$. Hence, by § 125 its sum is

$$s_\infty = \frac{\frac{p}{1+r}}{1-\frac{1}{1+r}} = \frac{p}{r}. \tag{11}$$

EXERCISES

1. If money is worth 5%, what is the cash equivalent of a yearly income of \$1500? *Ans.* \$30,000.

2. What is the value of a farm which returns an income of \$750 per year, continued indefinitely, money being worth 6%? *Ans.* \$12,500.

3. What is the capitalized value of a fixed salary of \$3000 a year, with money at 6%? *Ans.* \$50,000.

4. A railroad company pays a flagman \$1200 per year. What sum could the company afford to pay to install automatic machinery to do away with the services of the flagman, with money at 4%? *Ans.* \$30,000.

5. Is farm land which rents for \$9.00 per acre selling too high at \$250 per acre? What would be a reasonable rate of interest to allow in this case?

135. Depreciation. Depreciation is the decrease in value of any perishable article which results from use and from advancing age. It may result also from progress in the arts and in methods of manufacture which renders an article obsolete, or out of date. Articles may become inadequate, and may have to be replaced, in the course of time, by others better adapted to the requirements. Thus, broadly speak-

ing, depreciation includes wear and tear, obsolescence, and inadequacy.

Annual depreciation is the annual theoretical decrease in value, expressed in money.

136. Methods of Determining Depreciation Charges. Various methods have been advocated to determine how much shall be charged to depreciation for a given article, or plant, or machine. It is convenient to measure depreciation in terms of time, notwithstanding the fact that it does not necessarily depend wholly upon the passage of time. Periods of time are merely a convenient way of comparing the rapidity of depreciation with the rapidity of other occurrences. In §§ 137–139 we shall consider three methods of computation.

137. The Straight Line Method. This method proceeds upon the theory that the annual depreciation of an article is *uniform* throughout its probable period of life. For example, if a harvester costs \$120, and has a probable life of 10 years with no salvage value at the end of that time, the annual depreciation is merely one tenth of \$120. If the salvage value is \$15, then the annual charge for depreciation would be $(\$120-\$15)/10=\$10.50$.

Let $x=$ the annual charge for depreciation on an article,

$V=$ its value or cost,

$V'=$ its salvage value after a period of n years.

Then, by the straight line method, we have

$$x=\frac{V-V'}{n}. \tag{12}$$

EXERCISES

1. Show that all considerations of interest are disregarded in the straight line formula for estimating the depreciation charge.

2. Draw a graph to show the value of the harvester of § 137 at various times after its purchase. Why is (12) called the straight line formula?

3. A house is worth \$8000 and its estimated life is 40 years. Allowing \$400 salvage value, what should be allowed each year for depreciation, using the formula (12)? Draw a graph.

4. What is the life of a wagon which depreciates about 5% of its original value each year? Draw a graph. *Ans.* 20 years.

5. A Minnesota farm bulletin shows the following average depreciation of machinery, in per cents of the original cost.

Machine	% Depreciation	Machine	% Depreciation
Threshing Outfit .	12.0	Gang Plows . .	7.40
Hay Loaders . .	11.78	Gasoline Engines	7.35
Manure Spreaders	11.67	Corn Cultivators	7.23
Corn Binders . .	10.03	Heavy Horses .	6.17
Harrows . . .	8.72	Wagons	4.89
Grain Binders . .	7.91	Fanning Mill . .	4.58

Determine the life of each machine by the straight line formula, estimating the salvage value at 5% of the cost in each case.

Estimate the value at the end of 6 years of a corn harvester costing originally \$125.

6. What is the annual charge for depreciation and what is the probable life of a hay loader that costs \$110, and that may be sold for \$30 after 7 years' usage?

7. If the cost of a machine is \$100, its probable life 20 years, and its salvage value zero, what is the annual charge for depreciation, using formula (12)?

Construct a graph for this information, using time in years on the horizontal axis, and depreciation in dollars on the vertical axis.

Construct a second graph, using time as before, but actual value of the machine on the vertical axis.

138. The Reducing Balance Method. This plan deducts for depreciation, at the end of each year, a certain constant percentage of the value of the article at the beginning of that year. This value itself decreases each year. Thus if an article costs \$100 and if the depreciation each year is 10%

of the value at the beginning of that year, its value at the end of the fourth year is found as follows:

10% of \$100 = \$10;
hence first balance = \$100 − \$10 = \$90;
10% of \$90 = \$9;
hence second balance = \$90 − \$9 = \$81;
10% of \$81 = \$8.10;
hence third balance = \$81 − \$8.10 = \$72.90;
10% of \$72.90 = \$7.29;
hence fourth balance = \$72.90 − \$7.29 = \$64.61

This process may be continued till the balance reaches a given salvage value. It will be noticed that by this method the charge for depreciation is high in the first years of life of an article and comparatively low in the later years of its life.

139. The Reducing Balance Formula. Let x denote the constant fractional rate of depreciation which is to be deducted each year from the balance; V, the original investment; V', the salvage value; and n, the number of years before the salvage value is reached. Then, deducting x of V from V', we have

the value at the end of 1 yr. $= V - Vx = V(1-x)$;
the value at the end of 2 yr. $= V(1-x) - V(1-x)x = V(1-x)^2$;
the value at the end of 3 yr. $= V(1-x)^3$;

. . . . . .

the value at the end of n yr. $= V(1-x)^n$.

But this nth balance is the salvage value, V'. Thus

$$V(1-x)^n = V', \quad i.e. \tag{13}$$

$$x = 1 - \sqrt[n]{\frac{V'}{V}}, \tag{14}$$

and

$$n = \frac{log\ V' - log\ V}{log\ (1-x)}. \tag{15}$$

EXAMPLE 1. What is the value at the end of the fifth year of an automobile costing \$500, if depreciation each year is 15% of the value at the beginning of the year?

By (13), $V' = V(1-x)^n$

$= \$500(1-.15)^5$

$= \$500(.4436) = \$221.80$

EXAMPLE 2. In what time will an automobile costing \$500 depreciate to \$100, allowing 15% reduction each year on the value at the beginning of the year?

By (15), $$n = \frac{\log 100 - \log 500}{\log (1-.15)}$$

$$= \frac{2.0000-2.6990}{9.9294-10} = \frac{-(.6990)}{-(.0706)}$$

$= 9.9$ years $= 10$ years, approximately.

140. The Sinking Fund Method. By the sinking fund method, a fixed sum of money is set aside each year, and is allowed to accumulate at compound interest. It derives its name from the fact that it employs the principles ordinarily used in establishing sinking funds to liquidate indebtedness and for purposes of replacements.

Let us assume that a machine costs \$1000, that its probable life is 25 years, and that its salvage value is zero. The problem then is to find what sum of money must be set aside each year in order that at the end of the 25 years the fund will have accumulated to \$1000 at 5% compound interest. The annual fixed sum thus set aside in any year plus the interest accumulated in that year, represents the depreciation charge for that year. This problem has already been discussed and solved in § 133. In the preceding example the annual payment x is to be found that will accumulate to \$1000 in 25 years at 5% compound interest. By (9), § 133, the annual payment is

$$x = \frac{Vr}{(1+r)^n - 1}.$$

Hence, for the example just cited,

$$x = \frac{(1000)(.05)}{(1.05)^{25}-1} = \$20.952 = \$20.95 \text{ per year.}$$

EXERCISES

1. A horse 3 years old costs \$250 and at age 28 has depreciated to \$10 in value. Find the annual per cent of depreciation by the reducing balance formula.

2. Find the value of the horse of Ex. 1, by the reducing balance method, at age 5, at age 8, at age 13, at age 18, at age 25, using the per cent of depreciation found in Ex. 1. Draw a graph to represent these values.

Does this computation of depreciation agree with the facts for the usual horse, *i.e.* is depreciation of a horse uniform over a period of years? Is it large at any time? When is it largest?

3. Use the straight line method to discuss Exs. 1 and 2.

4. What is the life of a grain harvester costing \$240, allowing \$10 salvage value, and annual depreciation of 10%, using the reducing balance method? *Ans.* 30 years.

5. Find by the reducing balance method the value of the harvester of Ex. 4 at the end of 2 years, 5 years, 10 years, 15 years, 20 years, 25 years. Plot the graph.

Discuss these values, and decide where the depreciation is largest, where smallest, where it *should* be largest, etc.

When is this binder worth \$125, including salvage?

6. Use the straight line method to solve and discuss Exs. 4 and 5.

7. A farmer buys a manure spreader for \$120. After using it for 3 years, he sells it for \$75. Discuss the annual depreciation charge (*a*) by the straight line method, (*b*) by the reducing balance method.

8. What annual payment should be set aside by the sinking fund method to replace a kitchen stove in 15 years, if it is valued at \$75, and if money is worth 5%?

9. What amount should be set aside annually to replace an automobile that costs \$1500, at the end of five years, if money is worth 5%?

10. A farmer purchases an automobile for \$2000, a harvester for \$500, and a thresher for \$1000. He assumes that the automobile will last for 6 years, the harvester 10 years, and the thresher 12 years. How much should be set aside annually to repurchase these machines when worn out, not counting salvage, if money is worth 5%?

141. Summary of Interest Functions and Other Formulas. In the following formulas, n denotes the number of years or periods of time, and r denotes the rate of interest.

I. $A = p(1+r)^n$, the accumulation of $\$p$ compounded n years.

II. $p = \dfrac{A}{(1+r)^n}$, the present worth of $\$A$ due in n years.

III. $s_{\overline{n}|} = \dfrac{(1+r)^n - 1}{r}$, the accumulation of \$1 paid at *end* of each year for n years.

IV. ${}_{\overline{n}|}s = (1+r)\left[\dfrac{(1+r)^n - 1}{r}\right]$, accumulation of \$1 paid at the *beginning* of each year for n years.

For the accumulation of $\$p$ paid annually, multiply III and IV by p.

V. $a_{\overline{n}|} = \dfrac{(1+r)^n - 1}{r} \cdot \dfrac{1}{(1+r)^n}$, the present worth of an annual payment of \$1 continued for n years. For an annual payment of $\$p$, multiply by p.

VI. $\dfrac{1}{a_{\overline{n}|}} = \dfrac{(1+r)^n r}{(1+r)^n - 1}$, the annual income which \$1 will purchase.

VII. $\dfrac{1}{s_{\overline{n}|}} = \dfrac{r}{(1+r)^n - 1}$, the annual payment which will accumulate to \$1.

VIII. $s_{\infty} = \dfrac{p}{r}$, the present value of an annual payment of $\$p$ continued indefinitely.

IX. $x = \dfrac{V - V'}{n}$, *straight line depreciation* from V to V' in n years.

X. $x = 1 - \sqrt[n]{\frac{V'}{V}}$, *reducing balance depreciation* from V to V' in n years.

XI. $x = \frac{Vr}{(1+r)^n - 1}$, *sinking fund allowance for replacement* of the original value, V, in n years.

MISCELLANEOUS EXERCISES

1. Find the amount of \$1000 at 4% compound interest for 10 years.

2. Compute the amount of \$1000 at 4% compound interest for 10 years, compounded semiannually.

3. Compute the amount of \$1000 at 4% compound interest for 10 years and 6 months.

[HINT. $(1.04)^{10\frac{1}{2}}$ may be computed readily by logarithms, but the commercial practice is to compute the compound interest to the end of the last even year, and then find the simple interest on this amount for the remaining fraction of a year.]

4. What sum at compound interest for 6 years at 4% will accumulate to \$1000?

State this exercise in terms of present worth.

5. What rate at compound interest is a man receiving when \$1000 will amount to \$1935.28 in 15 years?

6. In what time will a sum of money at 5% compound interest double itself? *Ans.* 14.2 years.

7. Find the amount of \$1 at 5% compound interest for 1 year, 2 years, 3 years, 4 years, etc., to 10 years. Construct a graph showing this information.

8. Construct a graph showing the amount of \$1 for 10 years as the rate varies. Use 2%, 2½%, 3%, etc., up to 6%.

9. How much will a man have saved at the end of 10 years if he deposits \$60 semiannually in a savings bank, and interest is compounded at 5% semiannually?

10. What would be the cash value of a farm for which a man agreed to pay \$1000 a year for 10 years, if money is worth 6%? How much if money is worth 4½%? *Ans.* \$7360.09; \$7912.72

11. A tractor that costs \$1500 must be replaced in 10 years' time. If money can be accumulated at 5% interest, what sum should be set aside each year to meet the cost of a new one at the end of the time?

Ans. \$119.26

12. What annual income for 30 years will \$10,000 purchase, if money is worth $4\frac{1}{2}\%$? *Ans.* \$613.92

13. What annual income for 15 years will \$10,000 purchase, if money is worth $4\frac{1}{2}\%$? What, if money is worth 4%?

Ans. \$931.14; \$899.41

14. If money is worth 5%, compare the following offers for a farm:

(*a*) A rental of \$600 per year for 20 years, or a cash price of \$10,000.

(*b*) A rental of \$600 per year for 10 years plus \$700 a year for the following 10 years, or a cash price of \$12,500.

CHAPTER X

AVERAGES AND MIXTURES

142. Mixtures. In preparing many compounds, foods, fertilizers, concrete, etc., many different mixtures of substances or of two varieties of the same substance, are in common use.

The simplest case is that of mixing two varieties of the same substance that differ in percentage of strength of some important ingredient. As an illustration, we shall first show how to determine the proper amounts of different grades of milk to form milk of a desired standard of quality.

143. Dairy Problems. Milk is standardized according to its butter fat content. Such a standard milk may be obtained by adding milk of a higher per cent of butter fat when the per cent is too low or by the addition of skim milk when the per cent is too high. Cream is also used in the standardizing process. Water cannot be used, since it contains none of the other elements necessary in milk, such as sugar, casein, etc. The amount of water in milk is quickly determined by the specific gravity test.*

Ordinary legal milk is about a 3% milk, *i.e.* 3% of its weight is butter fat. Skim milk is theoretically a 0% milk, but it usually contains a small trace of butter fat, from .1% to .5%.

* See *gallon* in the Table of Equivalents, Table V, Appendix, where the weight of milk is shown.

EXAMPLE. It is desired to add skim milk to 800 lb. of milk containing 4.5% of butter fat to form a standardized milk containing 4% of butter fat. How many pounds of skim milk must be added?

SOLUTION. Let x be the number of pounds of milk after adding skim milk. Then the amount of butter fat is $.04\,x$; hence

$$.04\,x = (.045)(800) \text{ lb.} = 36 \text{ lb.},$$

or

$$x = \frac{36}{.04} = 900 \text{ lb.}$$

It follows that 100 lb. of skim milk should be added.

Still another method is given in § 148.

EXERCISES

1. How many pounds of skim milk must be added to 200 pounds of 6% milk to standardize it to 4% milk?

2. How many pounds of skim milk must be added to 500 pounds of 5.8% milk to standardize it to 3% milk?

3. How much skim milk must be removed from 600 pounds of 3.2% milk to standardize it to 4% milk?

4. Given 200 lb. of 4% milk. Show how to standardize it to 5% milk by removing skim milk.

5. How much 20% cream can you make from 50 lb. of 6% milk? How much is it worth if butter sells at 56¢ per pound — one pound of butter fat being sufficient for 116⅔% of its weight in butter?

6. In standardizing milk containing 3.5% butter fat to milk containing 4% butter fat, what per cent of the whole amount of milk is the skim milk removed?

7. How much each of 6% milk and skim milk should be mixed to obtain 120 pounds of 4% milk?

8. What are the proportions, and what are the amounts, of skim milk and of 5% milk required to obtain 50 pounds of 3% milk?

9. How many pounds of 5.2% milk must be added to 200 lb. of 3% milk to make 4% milk?

SOLUTION: Let x = the number of pounds of 5.2% milk to be added. Then

$$.04(200+x) = .03 \times 200 + .052\,x.$$

Solve this equation for x.

10. How many pounds of 3.1% milk must be added to 54 lb. of 5.3% milk to make 4% milk?

11. Find the proportions in which to mix 5% milk with 3.5% milk to make 4% milk. How many pounds of each should be taken to make 90 pounds of 4% milk?

12. How much 5.2% milk and skim milk containing .1% butter fat should be used to make 50 lb. of 3.9% milk?

13. How many pounds of .1% skim milk should be added to 5 gallons of 5.3% milk to produce 3.5% milk?

14. How much .1% skim milk must be removed from 780 lb. of 3% milk to raise the butter fat content to 4%?

15. I wish to standardize 26% cream to 20% cream by adding 4% milk. What are the proportions of cream and milk to be used?

16. How much 30% cream should be added to 500 lb. of 3% milk to make 5% milk?

17. What is the price per pound of butter fat when 20% cream sells for 150¢ per gallon?

144. Arithmetic Mean, Geometric Mean, Average. The ***arithmetic mean***, or ***simple average***, of several numbers is their sum divided by their number. Thus the arithmetic mean or average of 4 and 9 is $(4+9)/2=6\frac{1}{2}$, of a and b is $(a+b)/2$, of a, b, c is $(a+b+c)/3$. The arithmetic mean of the n numbers $a_1, a_2, a_3, \cdots, a_n$ is

$$A=\frac{a_1+a_2+a_3+\cdots+a_n}{n}. \tag{1}$$

For example, the average length of nine ears of corn whose measurements are 3″, 4″, 4″, 6″, 7″, 7″, 8″, 9″, 10″ is

$$A=\frac{3+4+4+6+7+7+8+9+10}{9}=6\tfrac{4}{9}''.$$

The ***geometric mean*** of *two* numbers is the square root of their product; of *three* numbers, is the cube root of their product. The geometric mean of n numbers $a_1, a_2, a_3, \cdots, a_n$ is the nth root of their product, *i.e.*

$$G=\sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdots a_n}. \tag{2}$$

For example, the geometric mean of 4 and 9 is $\sqrt{4\times 9}=6$; the geometric mean of 4, 9, 12 is $\sqrt[3]{4\times 9\times 12}=7.56$

The geometric mean of two numbers is very often called the ***mean proportional*** between the two numbers.

An ***average***, or a ***mean***, is *any representative type* which is used to describe a whole collection of values or objects. Thus the collection of nine ears of corn noted above is described by saying that the average length of ear is $6\frac{4}{9}''$.

EXERCISES

1. A man walks a mile at the rate of 5 mi. per hour and a second mile at the rate of 3 mi. per hour. To what average rate for the two miles is this equivalent?

[HINT. Average time per mile $= \frac{1}{2}(\frac{1}{3}+\frac{1}{5})$ hr. $= \frac{8}{30}$ hr. This is $3\frac{3}{4}$ mi. per hour.]

2. What is the average rate for two miles the first of which is traveled at the rate of a miles per hour and the second at the rate of b miles per hour?

Ans. $\dfrac{2\,ab}{a+b}$.

3. What is the average rate for three successive miles traveled at the rates of 15, 20, 25 mi. per hour? *Ans.* 19.$^+$ mi. per hour.

4. What is the average rate for four successive miles, the rates for which are 15, 20, 25, 30 mi., respectively? *Ans.* 21.$^+$ mi. per hour.

5. A farmer makes 8% on his investment the first year. Adding this to his capital, he makes 10% on the amount the second year, and he makes 15% the third year. What is the average rate of increase for the three years?

[HINT. Let r be the average rate and let p be the principal he starts with. Then the amount of p dollars at an average rate r for 3 years is $p(1+r)^3$. This must equal the actual amount he has at the end of 3 years. This equality may then be solved for r.]

6. What is the average rate of increase r of a certain capital if the annual rates are r_1, r_2, r_3, respectively?

145. Meaning and Use of an Average or Mean. We saw in § 144 that any representative type or individual is an average or mean in so far as it represents, in some way, the whole collection to which it belongs. When we are

concerned with groups of individuals we can seldom make a definite quantitative statement that surely applies to any given individual. But we can make statements that apply to the average of the group with a certainty that increases as the number of individuals increases. For example, the average healthy person 30 years old will live to be 67, though any given one of them may die sooner or later than that. The whole business of life insurance is based upon expectations of this kind, which are based upon the experience of life insurance companies with hundreds of thousands of people. Again, we could not be certain of the true characteristics of Jersey cows from a single cow or from a few of them, but after a study of measurable characters of a large group of them, very definite statements can be made.

To *describe* in a few words or symbols a whole group whose individuals we have measured, we put aside the measurements of any one individual and construct a single, intermediate number or characteristic. This gives some kind of *average*, which will be descriptive of the whole group.

146. Different Kinds of Averages. There are five different kinds of averages in common use for different purposes, viz.:

(1) the ***arithmetic mean*** (§ 144),
(2) the ***weighted arithmetic mean*** (§ 147),
(3) the ***geometric mean*** (§ 144),
(4) the ***mode*** (§ 150),
(5) the ***median*** (§ 149).

All of these are in general practical use. Such phrases as average rainfall, average rate of interest, average wage, average college student, all refer to different kinds of averages.

147. The Weighted Arithmetic Mean. A slight modification of the simple arithmetic mean is the ***weighted arithmetic mean.*** For example, the arithmetic mean of the lengths of a hundred ears of corn is merely the sum of these lengths divided by one hundred. However, these hundred measurements may be arranged in half-inch groups as follows.

Inches	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0	10.5	11.0
Frequencies or number of ears	1	2	2	2	3	4	6	7	8	10	10	14	13	10	5	2	1

The weighted arithmetic mean of these lengths is found by taking each *group length* times the *number* in the group. The sum of these products divided by the total number of ears is the average length, *i.e.*

$$\text{Av. length} = \frac{1\times 3+2\times 3.5+2\times 4+\cdots+13\times 9+10\times 9.5+\cdots+2\times 10.5+1\times 11.0}{100}$$

$$= 7.7''$$

In general, if m_1, m_2, m_3, $\cdots$, m_r represent the class marks (*length* of ear, *temperature*, *per cent butter fat*, etc.) and if w_1, w_2, w_3, $\cdots$, w_r the frequencies (*number* of ears, *gallons*, *pounds*, etc.), then

$$(3)\quad \textit{Weighted Arithmetic Mean} = \frac{w_1m_1+w_2m_2+\cdots w_rm_r}{w_1+w_2+w_3+\cdots w_r}.$$

Thus, *the weighted arithmetic mean is obtained by multiplying each mark by the corresponding frequency, and dividing the sum of these products by the total number in the collection.*

The principal advantage of the weighted arithmetic mean over the simple average is that it is more easily computed.

The weighted arithmetic mean is widely used in statistics, in mixing various compounds of different purities or of different temperatures, in determining results of civil serv-

ice examinations where each subject is weighted, in mixing concrete from substances of known voids as sand and gravel, in locating centers of gravity of masses w_1, w_2, w_3, $\cdots$, w_r lying along a straight line at distances m_1, m_2, $\cdots$, m_r, respectively, from a fixed point in that line (or lying in a plane at these distances from a fixed line), etc., etc.

EXAMPLE. Where, on a uniform bar whose own weight is neglected, is the center of gravity (balancing point) of weights of 1, 2, 3, 2, 2, 1 pounds, if they are placed at distances of 4″, 6″, 11″, 12″, 20″, 30″, respectively, from one end of the bar?

The required distance from that end is the weighted average

$$A = \frac{1\times4+2\times6+3\times11+2\times12+2\times20+1\times30}{1+2+3+2+2+1} = 13''.$$

148. Weighted Average of Grades of Milk. It is frequently necessary to know quickly the results of mixing grades of milk, or other substances of different purities or of different temperatures. Such problems may be stated in terms of weighted averages, as illustrated by the following example.

EXAMPLE. What kind of milk is a mixture of 3 gal. of 5.3% milk and 5 gal. of 3.1% milk?

This is a problem in *simple averages* (§ 144) if we allow the repetition of the 5.3% grade 3 times and of the 3.1% grade 5 times, *i.e.*

$$\text{Average} = \frac{(5.3+5.3+5.3)+(3.1+3.1+3.1+3.1+3.1)}{3+5}$$

$$= \frac{3(5.3)+5(3.1)}{3+5} = 3.9\% \text{ milk.}$$

Thus, generally, w_1 pounds of an m_1 milk and w_2 pounds of an m_2 milk will give an average milk

$$A = \frac{w_1m_1+w_2m_2}{w_1+w_2}. \tag{4}$$

Clearing (4) of fractions and collecting, we have

$$w_1(A-m_1) = w_2(m_2-A),$$

or

$$\frac{w_1}{w_2} = \frac{m_2-A}{A-m_1}. \tag{5}$$

Equation (5) gives the *ratio of the amounts* of milk in terms of the kinds of milk. This leads at once to the so-called ***parallelogram method of standardizing mixtures.*** This is merely a graphic form of interpreting equation (5) as follows.

Place the average in the center of the parallelogram, Fig. 93, and the marks of the substances concerned (per cent, strength, temperature, etc.), at the left-hand corners. Subtract diagonally, placing the remainders (taken positively) at the opposite right-hand corners respectively. These remainders are the ratios of the substances mixed which produce the given or the desired average.

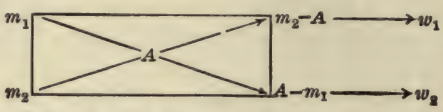
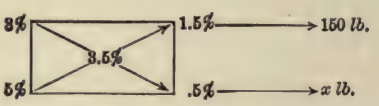


FIG. 93. — DAIRYMEN'S PARALLELOGRAM

EXAMPLE 1. A man has 150 lb. of 3% milk. How much 5% milk should be added to produce a 3.5% milk?

Let $x =$ the number of pounds of 5% milk to be added. Therefore, from the diagram



$$\frac{x \text{ lb.}}{150 \text{ lb.}} = \frac{.5}{1.5}, \text{ from which}$$

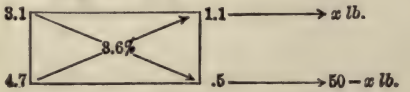
$$x = 50 \text{ lb.}$$

EXAMPLE 2. A farmer receives an order for 50 lb. of 3.6% milk. How many pounds each of 3.1% milk and 4.7% milk shall he take to fill the order?

Let $x =$ the number of pounds of 3.1% milk required and

$50 - x =$ the number of pounds of 4.7% milk required.

Therefore, from the diagram



we have

$$\frac{x}{50 - x} = \frac{1.1}{.5}, \text{ from which}$$

$$x = 34.4 \text{ lb.}$$

EXERCISES

1. A farmer mixes 8 gal. of 3% milk, 10 gal. of 2.9% milk, 5 gal. of 3.5% milk, 4 gal. of 5.3% milk, 6 gal. of 3.7% milk. What per cent butter fat is the mixture?

2. What kind of milk must be added to the mixture of Ex. 1 to make 36 gal. of 3.5% milk?

3. How much water must be added to a mixture of 3 pt. of 96% alcohol and 8 pt. of 78% alcohol to make the final mixture 60% alcohol?

[HINT. Let $x =$ the amount of water to be added.]

4. Masses of 4, 7, 9, 13 pounds are placed on a straight line at distances of $+2'$, $-3'$, $-4'$, $+5'$, respectively, from a given point O in that line. Where is the center of gravity with respect to this point?

Ans. $+1\frac{8}{33}'$ from the point.

5. Weights of 1, 2, 3, 4 pounds, respectively, are placed in order at the corners of a square $1'$ on each side. Find the center of gravity of this square, neglecting the weight of the square itself.

[HINT. Let two sides of the square intersecting at the one pound weight be the x- and y-axes as in the accompanying figure. Find the x-distance of the center of gravity as in the previous exercise. Similarly for the y-distance.]

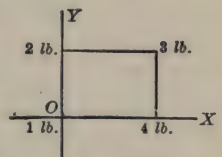


FIG. 94

6. Find the center of gravity of an equilateral triangle on which weights of 1, 2, 3 pounds, respectively, are placed at each corner.

7. A candidate receives the following marks in a civil service examination. In penmanship, weighted 1, 90%; in geography, weighted 2, 75%; in arithmetic, weighted 5, 70%; in U. S. History, weighted 3, 60%; in English, weighted 2, 80%. Compare his standing with the candidate who has, for the same subjects and weights, marks of 85% in penmanship, 77% in geography, 60% in arithmetic, 70% in U. S. History, and 70% in English.

[HINT. A weight of 1 in penmanship and of 2 in geography mean merely that geography is rated as twice as important as penmanship.]

8. Find the weighted arithmetic mean for the following table of measurements of 54 seeds.

Length in mm. . . .	1.56	1.58	1.59	1.60	1.61	1.62	1.63	1.64	1.65	1.66	1.67	1.68
Number of Seeds. . .	1	1	1	2	2	5	11	13	7	4	5	2

9. Solve Exs. 7–16, § 143, by the parallelogram method.

149. The Median. The *median* of a collection is a value or type such that half of the other values exceed it and half are below it. If the numbers are arranged in numerical order in a column, the number that is halfway down the column is the median. It may be found also by pairing off the largest and the smallest and repeating the process until only one or two is left. If two are left, the median may then be taken halfway between them.

EXAMPLE 1. The median of 1, 3, 3, 5, 6, 8, 10 is 5.

EXAMPLE 2. The median wage of 825 workmen is that of the middlemost of the men when they are arranged in numerical order with respect to wages, *i.e.* if 412 receive more than $2.77 and 412 receive less, then $2.77 is the median wage.

The advantage of the median as a representative type is the fact that it can be determined easily. Very large or very small values do not affect it. However, it gives no special importance to extreme values and may be after all far removed from the type.

150. The Mode. The *mode* is the *most frequent* value. In the set of values 3, 4, 4, 5, 6, the mode is 4 since it occurs twice and the others only once.

In a popular way, mode means fashion. Thus that type of citizen met oftener than any other is an average citizen. For a community that has three millionaires but all the other citizens in poverty, the arithmetic average for the financial condition of the community would give the impression that all the citizens were well-to-do, while in reality the *average citizen* is in poverty. The mode is useful in describing such a situation.

EXERCISES

1. Find the median length for the 100 ears of corn in the table of § 147.

2. Find the median length of seed for the seeds tabulated in Ex. 8, § 148.

3. Which mean is meant in the following cases? Average-sized apple; average coloration of Rome Beauty apples; average price of butter; average wheat production per acre in the United States; average rate of depreciation of farm machinery; mean daily temperature; normal rainfall; average student.

4. What per cent increase per year will double the attendance at a University in 10 years' time?

[HINT. Use $\sqrt[10]{2}$. Why? Compute this value by logarithms.]

5. What is the average annual rate of increase of wheat production, if it has increased from 525 millions of bushels to 775 millions in 10 years' time?

6. What kind of averaged age is it, if out of a total of 100,000 people alive at age 10, about one half, viz. 51,230, are living at age 64?

7. What is the mode for the table of measurements in Ex. 8, § 148?

151. Mixing Fertilizers. The use of commercial fertilizers is one method of returning to the soil plant foods needed by growing plants. These fertilizers contain nitrogen, phosphoric acid, and potash in various proportions for different crops and soils. It is quite often more satisfactory and much cheaper to buy the fertilizing materials and mix them on the farm.

A fertilizer labeled 2-8-4 contains 2% nitrogen, 8% phosphoric acid, and 4% potash. Nitrogen is obtained chiefly from nitrate of soda, dried blood, dried fish scrap, and cottonseed meal. Phosphoric acid (phosphorus) is obtained from ground bone, basic slag, ground phosphate rock, and acid phosphate. Potash is obtained from muriate of potash, sulphate of potash, and kainite. The usual composition of fertilizing materials is about as follows. Chilean nitrate of soda contains 15% nitrogen; acid phosphate contains about 16% phosphoric acid; muriate of potash contains about 50% potash. More complete tables of fertilizing materials will be found in the Appendix, Tables III and IV.

Filler, usually in the form of sand or dry dirt, is added to make the proper per cent and to aid in distributing the fertilizer evenly.

EXERCISES

1. A farmer wishes to make 1 ton of 2-8-4 fertilizer from Chilean nitrate, acid phosphate, and muriate of potash. (See § 151.) How much of each ingredient must he use and how much filler must he add?

[HINT. The nitrogen content is 2% of 2000 pounds = 40 pounds. This must be obtained from Chilean nitrate which is only 15% nitrogen. Therefore if x is the number of pounds of nitrate used, $.15\,x = 40$; hence $x = 267$ pounds.] *Ans.* Filler = 573 pounds.

2. Show how to make one ton of 3-10-3 fertilizer, which is used for raising corn or wheat. Calculate its cost at 3¢ per pound for Chilean nitrate, 1¢ per pound for acid phosphate, and 3¢ per pound for muriate of potash.

3. Show how to make a 4-9-4 fertilizer, used for raising potatoes, from the materials of Ex. 2. Calculate the cost per ton.

4. A farmer has a ton of fertilizer containing 30% dried blood, $62\frac{1}{2}$% acid phosphate, and $7\frac{1}{2}$% muriate of potash. How many pounds of acid phosphate must he add to this ton to raise the acid phosphate content of the mixture to 70%? *Ans.* 500 lb.

5. Compare a 2-8-2 fertilizer at $25.00 per ton and a 2-8-6 fertilizer at $30.00 per ton on the basis of the average cost per pound of *plant food* which they contain. What kind of average is this?

6. How many pounds of acid phosphate must be added to 400 lb. of a 2-6-4 fertilizer to convert it into a 2-8-4 fertilizer, if the acid phosphate contains 16% phosphoric acid? *Ans.* 100 pounds.

7. How many pounds of acid phosphate and how many of muriate of potash must be added to a ton of a 2-6-4 fertilizer to convert it into a 2-8-6 fertilizer?

8. A farmer mixes 500 pounds of a 2-9-5 fertilizer and 300 pounds of a 2-6-2 fertilizer. What ratios do the ingredients of the mixture have?

9. Find the ratios of the ingredients for 200 lb. of a 2-6-4 fertilizer mixed with 300 lb. of a 1-2-1 fertilizer.

In solving the following problems, use Tables III and IV, Appendix, which give the required amounts of fertilizing materials per acre to grow various crops, and which give the composition of these materials.

10. What quantities of dried blood (13% nitrogen) and bone meal (3% nitrogen, 24% phosphate) are needed for an acre of wheat ground which is strong in potash but which requires 12 lb. nitrogen and 20 lb. phosphate per acre?

SOLUTION: Let x = lb. dried blood required.
y = lb. bone meal required.

In x lb. dried blood there are $.13\,x$ lb. nitrogen, and in y lb. bone there are $.03\,y$ lb. nitrogen and $.24\,y$ lb. phosphate. Hence the total nitrogen is $.13\,x+.03\,y=12$ lb., and the total phosphate is $24\,y=20$ lb. Solving these two equations for x and y, we find $x=73$ lb., $y=83$ lb.

11. In Ex. 10, substitute nitrate of soda and phosphate of lime for dried blood and bone meal.

12. In Ex. 10, substitute corn for wheat.

13. A gardener has a supply of wood ashes and a limited amount of hen manure mixed with an equal weight of leaves. How much dissolved bone should he buy to make a complete fertilizer for an acre of beets?

SOLUTION:

Analysis of hen manure	2	− 2	− 1	%
Analysis of leaves	.7	− .15	− .3	%
Average for equal amounts	1.35	− 1.08	− .65	%
Analysis of dissolved bone	2	− 20	− 0	%
Analysis of wood ashes	0	− 1	− 5	%

The requirement for an acre of beets is

20 lb. nitrogen − 25 lb. phosphate − 35 lb. potash.

Let x = no. of lb. of mixed hen manure and leaves required per acre,
y = no. of lb. of wood ashes per acre,
z = no. of lb. of dissolved bone per acre.

Then the total nitrogen is

$$.0135\,x+\qquad .02\,z=20 \text{ lb.},$$

the total phosphate is

$$.0108\,x+.01\,y+.20\,z=25 \text{ lb.},$$

and the total potash is

$$.0065\,x+.05\,y\qquad =35 \text{ lb.}$$

Solving these equations, we find approximately,

$$x=1460 \text{ lb.},\ y=510 \text{ lb.},\ z=21 \text{ lb.}$$

14. A man wishes to raise an acre of potatoes. He has at hand 500 lb. farmyard manure and 100 lb. hen manure. These are to be mixed in this proportion when used with wood ashes to make a minimum complete fertilizer for the particular soil used. How many pounds of each are required?

15. Find the proportions of wood ashes and hen manure for an acre of celery on soil that requires no additional nitrogen.

16. A man has applied 4000 lb. of farmyard manure to an acre of tomato land. How much wood ashes, dried ferns, and hen manure, mixed in the proportions 4, 4, and 1, respectively, should he use to complete the fertilization?

17. Find a complete fertilizer made of acid bone and sulphate of ammonia for an acre of cabbage on clay land that requires no potash.

18. What quantities of floats, farmyard manure, and sulphate of potash are needed for raising an acre of onions?

152. The Mixing of Concrete. Concrete is usually a mixture of cement, sand, and gravel (c, s, g), in various proportions, depending upon the use to which it is to be put. Since there is a certain per cent of air space (***voids***) in sand and gravel, it is necessary to calculate the various amounts of c, s, g to make a given volume, say 1 cu. yd., of solid concrete. A common working rule is to allow $33\frac{1}{3}\%$ voids in the sand and 45% voids in the gravel or crushed stone.

EXAMPLE. How many cubic yards each of cement, sand, and gravel, mixed in the ratio 1-2-4 (*i.e.* 1 part cement to 2 parts sand to 4 parts gravel) are required to make one cubic yard of solid concrete, allowing $33\frac{1}{3}\%$ voids in the sand and 45% voids in the gravel?

Let x = the cubic yards cement,
y = the cubic yards sand,
z = the cubic yards gravel required.

Since the voids in the gravel are filled up in the finished concrete, the z cu. yd. of gravel count as only $.55\,z$ cu. yd. of solid concrete, and the y cu. yd. of sand count as only $\frac{2}{3}\,y$ cu. yd. of solid, finished concrete. There being no voids in the cement, we have the equations:

(1) $x+\frac{2}{3}\,y+.55\,z=1$ (cu. yd. of concrete);

(2) $y = 2x$ and $z = 4x$ from the ratios of the mixture.

Substituting the values of y and z from (2) in (1), we have

$$x + \frac{4x}{3} + 2.2x = 1.$$

Solving this equation we obtain

$$x = .22 \text{ cu. yd. cement,}$$

from which we find $y = .44$ cu. yd. sand, and $z = .88$ cu. yd. gravel.

153. Unit of Volume of Cement. Cement varies greatly in weight in different stages of compactness. Results show a variation from 86 to 123 pounds in the average weights of the same cement according as it was weighed *sifted* or *packed* in a barrel. The quantity of cement, in the proportion for mixing concrete, should therefore invariably be regulated by its weight. In case the proportions are by volume, then a definite weight or a definite number of packages of cement must be assumed as the ***unit volumes.***

As to the unit which should be selected for the volume of a cement barrel, the opinion of some fifty authorities (members of the American Society of Civil Engineers) differed greatly. An approximate average of all the figures suggested by them was that a barrel of cement should be 3.8(3.76) cubic feet, which corresponds to 100 pounds to the cubic foot. There are four sacks of cement in 1 barrel.

EXERCISES

Complete the following table giving the ingredients for 1 cu. yd. of solid concrete, allowing $33\frac{1}{3}$ % voids in the sand, 40 % in the gravel.

Number	Mixture	Cement		Sand	Gravel
		Cu. Yd.	Sacks		
1	1-2-3		7.2		
2	1-$2\frac{1}{2}$-5				
3	1-3-5				
4	1-3-6				
5	1-2-2				

6. Give the ingredients for a cubic yard of 1-2-3 concrete allowing 20% voids in the sand and 40% in the crushed stone.

7. Solve as in Ex. 6 for a 1-3-5 concrete.

154. Proportions of Cement, Sand, Gravel for Various Purposes. The following are the most commonly accepted proportions of cement, sand, and gravel, respectively, for various purposes.

(*a*) Rich: 1-1½-3 for columns and structural parts under very heavy stress.

(*b*) Standard: 1-2-4 for floors, beams, and columns under considerable stress.

(*c*) Medium: 1-2½-5 for walls, piers, sidewalks, etc., and not subject to great stress.

(*d*) Lean: 1-3-6 for heavy mass work under compression only.

155. Fuller's Rule. For determining more quickly the amount of the ingredients in a cubic yard of concrete, the following formulas, known as ***Fuller's Rule***, give rather good working results:

(5) $C = \dfrac{11}{c+s+g}$ = the barrels of cement in 1 cu. yd. of concrete,

(6) $S = \dfrac{C(s)(3.8)}{27}$ = the cubic yards of sand in 1 cu. yd. of concrete,

(7) $G = \dfrac{S(g)}{s} = \dfrac{C(g)(3.8)}{27}$ = the cubic yards of gravel in 1 cu. yd. of concrete,

where

c = the number of parts of cement = 1 always,

s = the number of parts of sand,

g = the number of parts of gravel.

To reduce barrels of cement to sacks, multiply the number of barrels by 4.

EXERCISES

Fill out the accompanying diagram according to Fuller's Rule and thus obtain some working rules.

Parts in the Formula	Sacks Cement	Cu. Yd. Cement	Cu. Yd. Gravel
1. 1-1½-3			
2. 1-2-4			
3. 1-2½-5			
4. 1-3-6			

5. Compare the amounts of material necessary for 1 cu. yd. of a 1-2½-5 concrete mixed according to Fuller's Rule and according to the example of § 152.

CHAPTER XI

GEOMETRY — MENSURATION

156. Right Triangles. Some elementary mensuration rules were stated in Chapter II. The purpose of the present chapter is to simplify what has been given, and to give geometric rules and theorems of more advanced character.

In a right triangle ABC, shown in Fig. 95, let the sides be denoted by a, b, c, in the usual manner (§ 15), and let the perpendicular h, let fall from C on the side c, divide c into the two parts m and n (Fig. 95). Then the following relations hold:

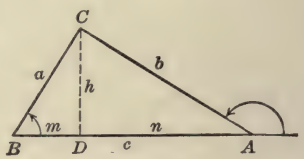


Fig. 95

(1) $a^2+b^2=c^2$,

(2) $h^2=mn$,

(3) $a^2=mc$, $b^2=nc$.

Each of these theorems may be stated in words by the student: (1) is the Pythagoras Theorem (§ 15); (2) and (3) may be obtained easily from the fact that the triangles ABC, ADC, and BDC are all similar; (1) is obtained by adding the two parts of (3).

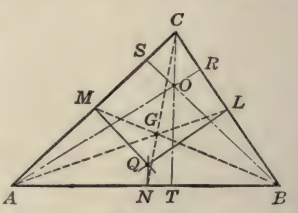


Fig. 96

157. Oblique Triangles. In any triangle ABC (Fig. 96), the following theorems hold.

An ***exterior angle*** made by any side and an adjacent side produced is equal to the sum of the two opposite interior angles.

The ***altitudes*** meet in a point O, called the ***orthocenter.***

The ***medians***, joining each vertex with the midpoint of the opposite side, meet in the ***center of gravity G.*** This point G trisects each median.

The ***perpendiculars*** erected at the midpoints of the sides meet in a point Q which is the ***center of the circumscribed circle.***

In any triangle O, G, *and* Q *lie in a straight line*, and G is two thirds of the way from O to Q.

The *bisectors of the angles* meet in a point P which is the ***center of the inscribed circle*** (Fig. 97).

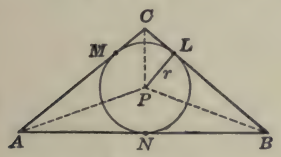


FIG. 97

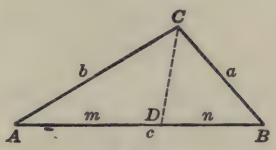


FIG. 98

The bisector of an angle divides the opposite side into segments which are proportional to the adjacent sides (Fig. 98).

158. Extension of the Pythagoras Theorem for Any Triangle. In any triangle ABC (Figs. 99, 100), if h is the altitude on b, and x is the projection of c on b, it is shown

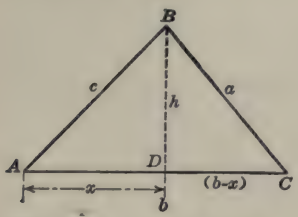


FIG. 99. — $a^2 = b^2 + c^2 - 2bx$

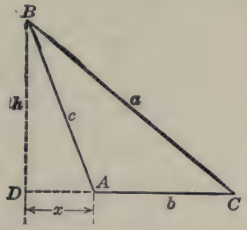


FIG. 100. — $a^2 = b^2 + c^2 + 2bx$

in plane geometry, by using the Pythagoras theorem on the two right triangles ABD and DCB, that

(4) $a^2 = b^2 + c^2 - 2\,bx$, *if* A *is acute,*

or

(5) $a^2 = b^2 + c^2 + 2\,bx$, *if* A *is obtuse.*

Solving these equations for x, we may write also

(6) $x = \dfrac{b^2 + c^2 - a^2}{2\,b}$, *if* A *is acute,*

or

(7) $x = \dfrac{a^2 - b^2 - c^2}{2\,b}$, *if* A *is obtuse.*

159. Computation of Altitudes. Area of a Triangle. If the three sides of a triangle are given, we may compute

(*a*) The ***projections*** of any side upon any other side (§ 158).

(*b*) The ***altitudes***. From Fig. 99 or 100, $h = \sqrt{c^2 - x^2}$ where x is known from (6) or (7), § 158.

(*c*) The ***area***. When the altitude has been found, the area follows at once. The formula for area of a triangle from § 15, viz.:

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

may be derived from § 158. For, from Figs. 99, 100,

$$h^2 = c^2 - x^2 = c^2 - \left(\frac{b^2 + c^2 - a^2}{2\,b}\right)^2 = \frac{4\,b^2c^2 - (b^2 + c^2 - a^2)^2}{4\,b^2};$$

or, factoring the numerator,

$$h^2 = \frac{(a+b+c)(b+c-a)(a-b+c)(a+b-c)}{4\,b^2}.$$

Let $a+b+c = 2\,s$. Then we have

$$a+b-c = 2\,s - 2\,c = 2\,(s-c),$$
$$a-b+c = 2\,s - 2\,b = 2\,(s-b),$$
$$-a+b+c = 2\,s - 2\,a = 2\,(s-a).$$

Therefore

$$h^2 = \frac{2\,s\,2\,(s-a)\,2\,(s-b)\,2\,(s-c)}{4\,b^2},$$

and

$$h = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}. \tag{8}$$

Hence the area, which is equal to $h \times \frac{b}{2}$, is also equal to

$$\textit{Area} = \sqrt{s(s-a)(s-b)(s-c)}. \tag{9}$$

EXERCISES

1. The sides of a triangle are 9, 12, 15. Find the segments of the longest side made by bisecting the angle opposite it.

2. The sides of a triangle are 12, 14, 16. Draw it to scale using 14 as the base and find

(*a*) The segments of the side 14 made by the bisector of the opposite angle.

(*b*) The projections of the sides 12 and 16, respectively, on the side 14.

(*c*) The altitude on the side 14.

(*d*) The area of the triangle.

3. Find the area of a triangular pasture whose sides measure 19, 32, 25 rods, respectively.

160. The Circle. To the elementary facts about circles given in §§ 19, 20, we may add the following theorems, which are usually proved in elementary geometry.

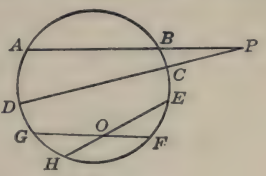


FIG. 101

I. *An angle between two secants is measured by one half of the difference of the intercepted arcs.*

Thus, we have for example $\angle APD = \frac{1}{2}$ (arc AD − arc BC), Fig. 101.

II. *An angle between two chords is measured by one half of the sum of the intercepted arcs.*

Thus, $\angle EOF = \frac{1}{2}$ (arc EF + arc GH), Fig. 101.

III. *If a tangent PT and a secant PAB be drawn from any exterior point P to a circle* (Fig. 102), *the square of the tangent PT is equal to the product of the two segments of the secant, PA and PB; i.e.*

$$(10) \qquad PT^2 = PA \cdot PB = PC \cdot PD$$
$$= \text{a constant.}$$

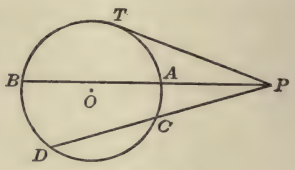


FIG. 102

This fact is of interest with reference to *visibility above the surface of the earth.* Let PB, Fig. 103, be a diameter of the earth, PA the distance above the earth's surface, and PT the distance to the horizon. Then

$$PA \cdot PB = PT^2.$$

Let $PA = k$; then $PB = k + 2r$. Also let $PT = d$. Therefore $d^2 = k(k + 2r)$.

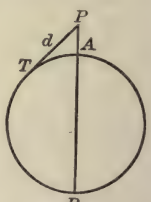


FIG. 103

Practically, k is small as compared with the diameter of the earth. Thus, $k + 2r$ may be written as $2r$. Hence

$$d^2 = 2kr, \quad \text{or} \quad k = \frac{d^2}{8000} \ (k \text{ and } d \text{ in miles}).$$

But k is usually measured in feet. Therefore let $k = \frac{h}{5280}$ and

$$k = \frac{h}{5280} = \frac{d^2}{8000}, \text{ from which}$$

$$h, \textit{ the height in feet} = \tfrac{5\,2\,8}{8\,0\,0}\, d^2 = \tfrac{2}{3}\, d^2, \text{ approximately.*}$$

Therefore, *the height in feet equals two thirds the square of the distance in miles.*

EXAMPLE. A lighthouse is 240 feet high. How far is its light visible at sea?

SOLUTION: We have $h = 240$; hence

$$240 = \tfrac{2}{3}\, d^2, \quad \text{or} \quad d = 19 \text{ miles.}$$

* See Exercise 9, § 79.

161. Compound Curves. Arcs AB and AC of two circles, which are tangent to each other at A, Figs. 104–105, form a ***compound curve.*** The point of tangency A is the ***transition point*** from one circle to the other.

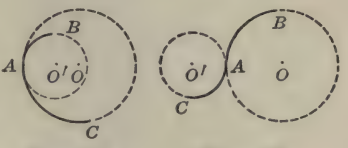


Fig. 104 Fig. 105

Compound curves are used in the construction of railroads and highways where the track must be made to conform to the physical features of the land.

162. The Area of a Segment of a Circle. In § 22, mention was made of the area of a segment of a circle. In Fig. 106, let

h = height of a segment,

r = radius of the circle.

Then $OS = r - h$, and the angle SOP is given by

$$\cos SOP = \frac{OS}{OP}. \tag{11}$$

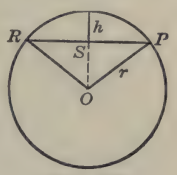


Fig. 106

The area of the sector $OPQR$ and of the triangle OPR may now be found. The required area of the segment is their difference.

EXERCISES

1. Find the diameter of a broken wheel a chord of which measures 16″ and the height of the corresponding arc is 4″.

2. The chord of an arc is 24″ and the chord of half this arc is 13″. Find the diameter of the circle.

3. Show that the following construction, Fig. 107, will divide a circle into three parts whose areas are equal.

Trisect the radius at B and C. Draw a semicircle on OA as diameter. Erect perpendiculars BR, CP meeting the semicircle at R, P. With centers at O, draw the required circles with radii OP and OR.

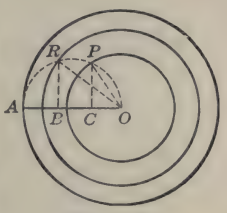


Fig. 107

4. A line segment approximately equal to the circumference of a circle may be constructed as follows.

Draw a diameter AD, Fig. 108, and $COT = 30°$. Draw CT perpendicular to AD. From D draw DR perpendicular to AD and equal in length to three times the diameter. Then CR is approximately equal to the circumference.

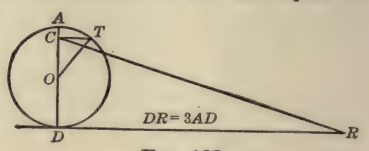


FIG. 108

Show that the error in the circumference, constructed in this way, is about 1 in 20,000.

Use the formula of § 160 to solve the following exercises.

5. How high must an aeronaut rise above the earth in order to see a city 50 miles away? How high in order to see 100 mi.?

6. How far can a man 6 ft. tall see when standing on a level plain?

7. The lookout in the rigging of a vessel is 100 feet above the water. How far can he see a beacon light known to be 150 feet high?

8. The Woolworth building in New York City is 750 feet high. How far can it be seen by the lookout on a vessel, if the lookout is 75 feet above the water?

9. The curve ABC of a railroad track is composed of two circular arcs AB and BC, as in the accompanying figure. The center O of BC is on the radius PB of AB. Prove that ABC is a *compound curve, i.e.* that AB, BC are arcs of tangent circles.

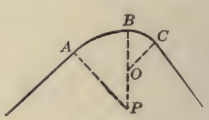


FIG. 109

10. The track of a switch $AOBC$ from a straight railroad track AD, as in the accompanying figure, is laid out as follows. Arc AO is constructed with center M, where AM is perpendicular to AD. MO is produced to N so that $MO = ON$ and arc OB with center N is drawn.

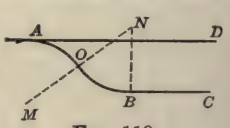


FIG. 110

Show that AOB is a *compound curve.*

11. By sawing off 4″ of the diameter of a log 12″ in diameter what part of the area is sawed?

12. The height of a segment of a circle is 3″ and its base is 10″. Find the radius and the area of the corresponding circle, and the area of the segment. *Ans.* $r = 5\frac{2}{3}$″.

163. The Ellipse. *An ellipse is formed when a point moves so that the* SUM *of its distances from* TWO FIXED POINTS is constant.

The two fixed points are called ***foci***, F and F' in Fig. 111. That diameter AA' on which the foci are located is called the ***major axis*** and is taken equal to $2a$, *i.e.* $OA = a$. Similarly the other axis, called the ***minor axis***, is perpendicular to AA' at O and is taken equal to $2b$, *i.e.* $OB = b$ (Fig. 111).

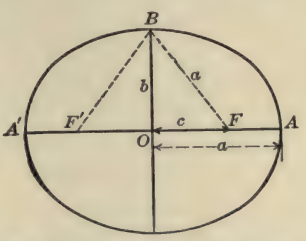


FIG. 111. — ELLIPSE WITH AXIS AND FOCI

An ellipse may be constructed by attaching the ends of a string to two pins fastened in a board or a piece of stiff cardboard and using the string as a loop, keeping it stretched by means of a pencil and allowing the pencil to move. Of course the length of the string is constant — the ends being fastened. When the pencil is at B, Fig. 112, and pins at F and F', the string takes the position FBF'. When the pencil is at A', the string reaches from F to A' and back to F'. With the pencil at A, we deduce $AF = A'F'$. In either case, then, the length of the string is equal to the length of the major axis, $2a$. Thus also $FB = F'B = a$.

The ***perimeter of an ellipse*** is given approximately by the formula

(12) $$P = \pi(a+b).*$$

The ***area of an ellipse*** is given exactly by the formula

(13) $$A = \pi ab.$$

---

* A more exact formula for the perimeter of an ellipse is

$$P = \pi(a+b)\left[1 + \frac{1}{4}\left(\frac{a-b}{a+b}\right)^2 + \frac{1}{64}\left(\frac{a-b}{a+b}\right)^4 + \cdots\right].$$

The formula $\pi(a+b)$ is very nearly correct if $a - b$ is small; *i.e.* if the ellipse is nearly round.

EXAMPLE. Construct an ellipse by attaching the ends of a string 4′ long to two pins $3\frac{1}{2}''$ apart fastened in a cardboard. By measurement, the major axis, $2a = 4''$, $FB = OA = 2''$, and $OF = 1\frac{3}{4}''$. Thus

$$OB = \sqrt{2^2 - (1\tfrac{3}{4})^2} = .97 = 1''$$

approximately. Hence $a = 2''$, $b = 1''$, and we have

$$\text{Area} = \pi ab = (\tfrac{22}{7})(2)(1) = 6.3 \text{ sq. in.}$$

$$\text{Perimeter} = \pi(a+b) = (\tfrac{22}{7})3 = 9.4''.$$

Many things are elliptic in shape or in cross-section. For example, sections of cupola ventilators placed on the roof of a barn are ellipses. Many photograph mountings and interior decorations are elliptic. Many arches in buildings, bridges, etc., are elliptic. Heat-conducting pipes for furnaces have many sections that are elliptic.

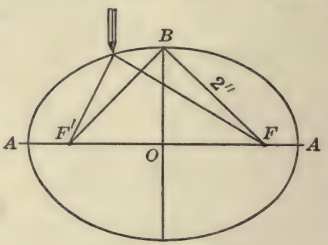


FIG. 112

The paths or orbits of the earth and the other planets about the sun are rather round ellipses with the sun at one focus. Hence the earth is nearer the sun during a part of the year and it moves faster then because of the increased attraction. When is the earth nearest the sun? How much nearer is it at that time?

EXERCISES

1. Find the area and the perimeter of an elliptic photograph 6″ long and 4″ wide.

2. A carpenter wishes to construct a frame for supporting the brick-work while laying up an arch over a grate, the arch to be half of an ellipse 46″ wide with a rise of 11″. Draw the ellipse to scale and show how to locate the foci. Find their distance from the center of the ellipse.

3. Compute the average velocity of the earth per second in its orbit about the sun, having given that its nearest and farthest distances from the sun are 90 millions and 93 millions of miles respectively.

4. Elliptic gears will *mesh* with each other when revolving about their foci.

At what distance from the end of the major axis should the hole for the center of revolution be drilled in an elliptic gear whose axes are 8″ and 10″?

Give an account of the elliptic gears used on a haypress to operate the plunger.

5. How large an opening — in area and perimeter — is necessary to install a circular ventilating cupola 36″ in diameter on a roof which has a pitch of 5 in 24?

How many 7″ circular intakes will be necessary to equal approximately the area of the opening in the roof?

164. Truncated Prism. A truncated prism is the part of a prism included between the base and a section made by a plane *oblique* to the base.

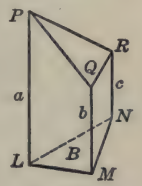


Fig. 113

The volume of a right truncated triangular prism, Fig. 113, is equal to the product of one third the sum of its lateral edges by the area of the base to which those edges are perpendicular, *i.e.*

(14) $$\textit{Volume} = \tfrac{1}{3}(a+b+c) \times \textit{base}.$$

The same formula holds for any triangular truncated prism (Fig. 114) if the *base* be replaced by the area of a section perpendicular to the edges.

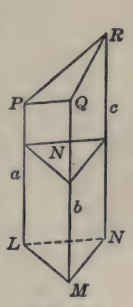


Fig. 114

Example. Corn is piled into the corner and along the side of a square building. It measures 4′ high and 6′ × 21′ on the floor and 15′ along the top crest. Find the volume of the pile.

The pile takes the shape of a right truncated triangular prism. Accordingly its volume is

$$\tfrac{1}{3}(15+21+21) \cdot \tfrac{1}{2}(4 \times 6) = 228 \text{ cu. ft.}$$

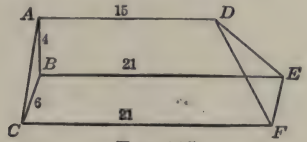


Fig. 115

EXERCISES

1. In excavating on the side of a hill for a building the following measurements were taken. From the surface to the bottom level of the cut 12′, 9′, 8′, 6′, respectively, at the four corners of the rectangular bottom which is to be 30′ × 40′. Find the amount of dirt to be removed.

[HINT. The shape of the required excavation will be that of two truncated right triangular prisms.]

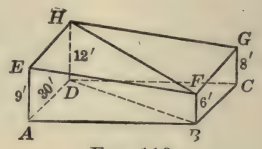


FIG. 116

2. What is the volume of the part of a rectangular barn under the roof and above the eaves if the barn measures 30′ × 60′ and the *hip roof* has a *pitch* of 1 in 3 on sides and ends?

Find also the angle made by the roof with the horizontal.

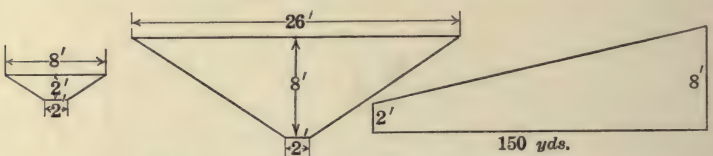


FIG. 117. — VERTICAL SECTIONS OF A DITCH

3. Find the number of cubic yards of gravel in a pile 8′ wide and 10′ long if the sides and ends have a *slope* of 2 in 5.

4. A cut for a ditch has the dimensions shown in Fig. 117, which shows the longitudinal vertical section and the vertical cross-sections at each end. Find the volume of the earth to be removed.

165. The Sphere. Definitions. A section of the sphere made by a plane through the center of the sphere is called a ***great circle;*** if the plane does not pass through the center of the sphere the section so made is a small circle. ***Parallels of latitude*** on the surface of the earth are small circles. The ***meridians***, which pass through both north and south poles, are great circles.

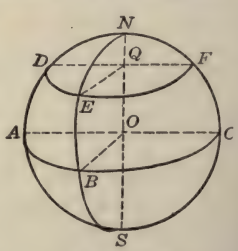


FIG. 118

A ***segment of the sphere*** is a portion of the volume of a sphere included between two parallel planes. It thus has two circular bases. In case one plane is tangent to the sphere, one base reduces to a point. The segment is then called a segment of one base. In Fig. 118, the part of the sphere above the plane DEF is a segment of one base; that between DEF and ABC is a segment of two bases.

A ***zone*** is a portion of the *surface* of a sphere included between two parallel planes (Figs. 118, 119).

The ***altitude*** of the segment or of the zone is the perpendicular distance between the planes.

166. Volume of a Spherical Segment. Area of a Zone.

(15) $$\textit{Volume} = \tfrac{1}{6}\pi h(3r_1^2 + 3r_2^2 + h^2),$$

where h denotes the altitude of the segment, and r_1, r_2 denote the radii of the bases, respectively.

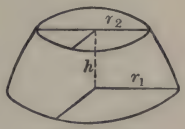


FIG. 119

(16) $$\textit{Lateral Area of Zone} = 2\pi rh,$$

where r denotes the radius of the sphere.

EXAMPLE. A sphere 8″ in radius is cut by two parallel planes, one passing 2″ from the center, the other 6″ from the center. Find the volume of the segment and the area of the zone between the two planes.

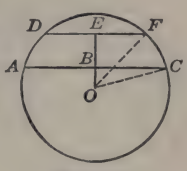


FIG. 120

The lateral area of the zone is

$$2\pi rh = 2\pi(8)(4) = 64\pi = 201 \text{ sq. in.}$$

To find the volume, the radii r_1, r_2 must be found. From the accompanying figure

$$r_1 = BC = \sqrt{\overline{OC}^2 - \overline{OB}^2} = \sqrt{8^2 - 2^2} = \sqrt{60}.$$

$$r_2 = EF = \sqrt{\overline{OF}^2 - \overline{OE}^2} = \sqrt{8^2 - 6^2} = \sqrt{28}.$$

$$\text{Volume} = \tfrac{1}{6}\pi h(3r_1^2 + 3r_2^2 + h^2) = \tfrac{1}{6}\pi 4(180 + 84 + 16) = 586 \text{ cu. in.}$$

EXERCISES

1. The diameter of a sphere is 6 feet. Find the surface of a segment whose height is 2 feet.

2. The radius of a sphere is 3 feet. Find the height of a zone whose area is equal to that of a great circle.

3. Find the area of a zone of one base whose radius is 2 feet and the height of the zone is 1 foot.

4. Find the volume of an apple whose circumference is 12 in.

5. Find the volume and the surface of a spherical segment, if the radii of its bases are 6′ and 8′, respectively, and its height is 3′.

6. How many pints of water can be put in a wash-basin in the shape of a segment of a sphere, if the distance across the top is 12 inches and the greatest depth is 4 inches?

7. What is the volume and the entire surface of a dishpan in the form of a segment of a sphere, if the distances across the top and bottom are, respectively, 20″ and 16″, and the depth is $5\frac{1}{2}$″?

8. Compute the capacity of the milk-can shown in the diagram if it has the following dimensions, expressed in inches, $D=13''$, $d=7''$, $a=1''$, $b=4''$, $h=16\frac{1}{2}''$.

9. How far from a sphere of radius 6″ must a light be placed so that one sixth of the surface of the sphere will be illuminated?

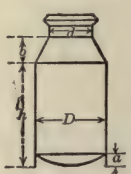


FIG. 121

167. Ellipsoid. The *ellipsoid* may be defined as the solid whose section by any plane is an ellipse, Fig. 122.

(17) $\qquad Volume = 4/3\ \pi abc,$

where a, b, c are the semi-axes.

FIG. 122. — ELLIPSOID

168. Paraboloid of Revolution. This is a solid generated by revolving a portion of a parabola about its axis, Fig. 123.

(18) $\qquad Volume = \pi r^2 h/2,$

where h is the height of the paraboloid and r is the radius of the base.

The surface of such a paraboloid is the usual reflecting surface for headlights, etc.

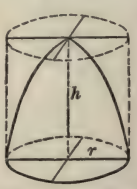


FIG. 123. — PARABOLOID

169. Anchor Ring or Torus. A torus or ordinary anchor ring is generated by a circle of radius r revolving about an axis in its plane (Fig. 124). If c is the distance of the center of the circle from the axis of rotation, and r is the radius of the circle, the volume and the surface of the anchor ring are given by the formulas

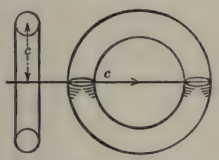


FIG. 124. — ANCHOR RING

$$\text{(19)} \qquad \textit{Volume} = 2\,\pi^2 cr^2,$$

$$\text{(20)} \qquad \textit{Surface} = 4\,\pi^2 cr.$$

170. The Prismoid Formula. A solid is called a ***prismoid*** if it has for bases any two polygons in parallel planes and for faces quadrilaterals or triangles, Fig. 125. The altitude of a prismoid is the perpendicular distance between the planes of its bases. The mid-section of a prismoid is that section made by a plane parallel to and midway between its bases. The mid-section bisects the altitude and all of the lateral edges.

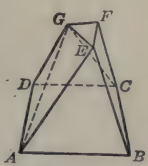


FIG. 125. — A PRISMOID

The volume of any prismoid is given by the following simple formula

$$\text{(21)} \qquad \textit{Volume} = (b + B + 4\,M)\frac{h}{6},$$

where h denotes the altitude of the prismoid, b and B denote the areas of the two bases, and M denotes the area of the mid-section.

If the area of one base, b, is zero, that base consists of a single point or a line, and the prismoid resembles a *pyramid*, a *cone*, or a *wedge*. For such a solid, $M = B/4$, and the formula becomes

$$V = B\frac{h}{3}.$$

If the bases, b and B, are equal, the prismoid resembles a *prism* or a *cylinder*, and the formula becomes $V = h \times B$.

A *frustum* of a pyramid or of a cone is a prismoid for which

$$\sqrt{M} = \frac{\sqrt{b}+\sqrt{B}}{2},$$

i.e.

$$4M = b + B + 2\sqrt{bB}$$

since the *mid-section of a frustum* is similar to and midway between the bases*; and the formula becomes

$$V = (b + B + \sqrt{bB})\frac{h}{3}.$$

The prismoid formula applies to the volume of a sphere, where $b = B = O$, and M is the area of an equatorial section of the sphere.

EXERCISES

1. Compute the volume of a football whose shape is an ellipsoid of semi-axes $3\frac{1}{2} \times 3\frac{1}{2} \times 5\frac{1}{2}$.

2. Compute the solid contents in cubic inches of an egg whose larger end is a semi-ellipsoid whose semi-axes are $\frac{7}{8}''$, $\frac{1}{16}^{8}''$, and $\frac{1}{16}^{8}''$, and whose smaller end is a semi-ellipsoid whose semi-axes are $\frac{1}{8}''$, $\frac{1}{16}^{8}''$, and $\frac{1}{16}^{8}''$.

3. Find the volume of a reflecting surface whose form is a paraboloid, if its height is 5″ and the radius of the base is $3\frac{1}{2}''$.

4. The cross-section of a solid wrought-iron ring is a circle of 3″ radius. The inner radius of the ring is 18″. Find the surface and the volume of the ring.

5. Find the surface and volume of a life preserver whose shape is an anchor ring, formed by revolving a circle of a radius 3″ about an axis 13″ from the center of the circle.

What weight of water would be displaced by such a preserver if it is submerged?

---

* If s and s' are corresponding sides of b and B, then $(s + s')/2$ is the corresponding side of M. Therefore,

$$\frac{S}{(s+s')/2} = \frac{\sqrt{b}}{\sqrt{M}}, \text{ and } \frac{S'}{(s+s')/2} = \frac{\sqrt{B}}{\sqrt{M}}.$$

Adding these two equations and reducing, we find

$$\frac{2}{1} = \frac{\sqrt{b}+\sqrt{B}}{\sqrt{M}}.$$

6. A barrel made of circular staves has an end diameter of 18″ and a bung diameter of 22″. It is 34″ high. What is its capacity in bushels, and in gallons?

[HINT. Use (8), § 31.]

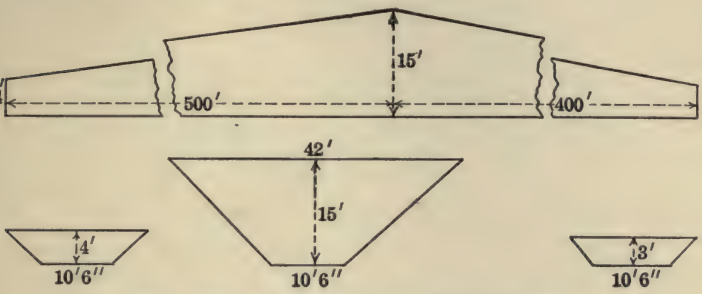


FIG. 126. — SECTIONS OF A RAILROAD CUT

7. If the barrel of Ex. 6 is standing on end, and if it is partly filled with a liquid to a depth of 24″, find the amount of the liquid in gallons.

[HINT. Calculate the amount of the contents above the half-barrel by the prismoid formula.]

8. A railroad cut has the dimensions shown in Fig. 126, which shows the vertical section and three cross-sections. Find the number of cubic yards of earth to be removed in this cut.

9. Find the number of cubic yards of concrete in a pier having the dimensions as shown in Fig. 127, the bases being rectangles with semicircles.

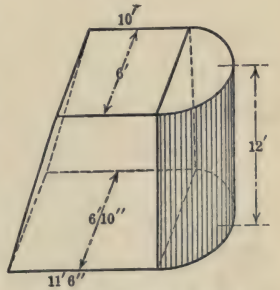


FIG. 127. — A PIER

CHAPTER XII

SOLUTION OF OBLIQUE TRIANGLES

171. Construction of Oblique Triangles with Ruler and Compass. A triangle consists of six parts — three sides a, b, c, and three angles A, B, C, respectively, opposite these sides. These parts are related as follows.

I. $A+B+C=180°$.

II. *The sum of any two sides is always greater than the third side.*

III. *The larger side is opposite the larger angle, i.e.* if $a>b$, then $A>B$, etc.

If any three parts of a triangle are given, including always at least one side, the triangle may be constructed. The following cases arise.

CASE I. GIVEN ONE SIDE AND TWO ANGLES.

Let c be the given side, and let A and B be the given angles (Fig. 128). Take $EF=c$, Fig. 128, and construct $\angle A$ at

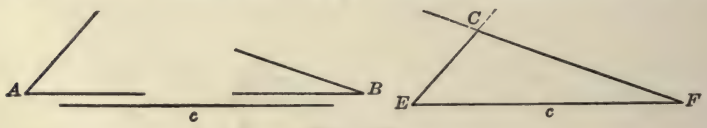


FIG. 128

E and $\angle B$ at F, on the same side of c. Produce the sides of $\angle E$ and $\angle F$ till they meet at C. Then EFC is the required triangle.

The third angle may be found, before constructing the triangle, from theorem I. If this angle is measured with a

protractor after the construction is finished, the accuracy of the construction may be judged.

CASE II. GIVEN TWO SIDES AND THE INCLUDED ANGLE.

Let b and c be the two given sides and ∠ E their included angle (Fig. 129).

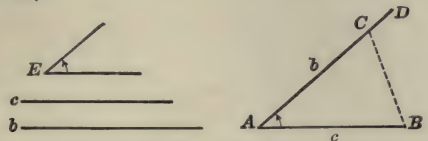


FIG. 129

Lay off $AB = c$, Fig. 129. At A, construct $\angle BAD = \angle E$. On AD take $AC = b$ and draw CB. Then $\triangle ABC$ is the required triangle.

CASE III. GIVEN THE THREE SIDES.

Let the three sides be a, b, c (Fig. 130).

Take BC equal to a, Fig. 130. From B as a center, with a radius equal to c, describe an arc. From C as a center, with a radius equal to b, describe an arc intersecting the

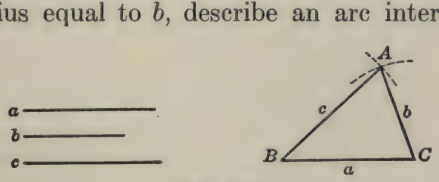


FIG. 130

other arc at A. Draw AC and AB. Then BAC is the required triangle.

CASE IV. AMBIGUOUS CASE. GIVEN TWO SIDES AND AN ANGLE OPPOSITE ONE OF THEM.

Let a and b be the given sides and A the angle opposite the side a (Fig. 131).

Different constructions are possible according as $a < b$, $a = b$, $a =$ *the altitude on c*, or $a <$ *the altitude on c*.

If $a<b$ and $\angle A$ ***acute***, there may be two constructions. Lay off a line AE, Fig. 131. At A, construct the given angle A and lay off $AC=b$. From C as center, with a

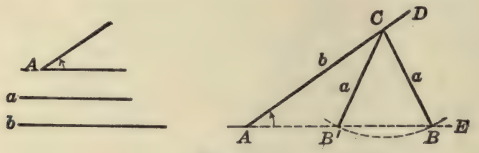


FIG. 131

radius equal to a, describe an arc intersecting AE at B and B'. Draw CB and CB'. Both triangles ACB and ACB' fulfill the conditions; there are two possible constructions.

If $a<b$ and $\angle A$ ***obtuse***, there is only one construction. The constructions for $a=b$, $a<b$, $a\leqq$ the altitude on c, are left as exercises for the student.

This case is known as the ***ambiguous case***. Why?

EXERCISES

1. Given angles of 50°, 75° and a side 2″ long in a triangle; construct the triangle, using rule and protractor. What is the third angle? *Measure* the other two sides.

2. Given $A=40°$, $C=110°$, and $c=2.5''$ in a triangle; construct the triangle. Find B and measure a and b.

3. Construct triangles to the scale 1 rod $=\frac{1}{8}''$, having given $a=14$ rods, $b=10$ rods, $B=30°$.

Construct, if possible, triangles having given

4. $a=14$, $b=6$, $B=30°$.

5. $a=14$, $b=16$, $B=30°$.

6. $a=14$, $b=6$, $C=30°$.

7. $a=14$, $b=6$, $c=16$.

172. Solution of Oblique Triangles. To *solve* a triangle is to *compute* the unknown parts when any three parts are given, at least one of which must be a side. This is one of the chief uses of trigonometry. An aid to such a computation is the construction of the given data to scale (§ 171).

The different cases that arise are those of § 171. If the given data are not sufficient, for example, if $a=10$, $b=6$, $c=3$, the triangle is not solvable. (See II, § 171.)

Oblique triangles can always be solved by dividing the triangle into two right triangles by drawing an altitude, and solving the resulting right triangles by previous methods. (See §§ 37–41.) However, this process is not always the most direct. The relations given in §§ 173–174 will aid greatly in the solution of the different cases.

173. The Law of Sines. *The sides of a triangle are proportional to the sines of the opposite angles,* i.e.

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}. \tag{1}$$

PROOF. Let ABC be any triangle, Fig. 132, and let p be the altitude on b. Then in all cases we have

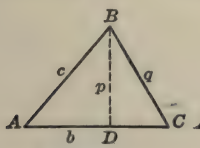


FIG. 132 *a*

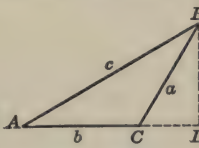


FIG. 132 *b*

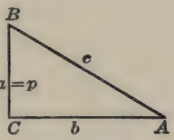


FIG. 132 *c*

$$\sin A=\frac{p}{c},\quad \text{and}\quad \sin C=\frac{p}{a}. \tag{2}$$

By dividing, we have

$$\frac{\sin A}{\sin C}=\frac{a}{c},\quad \text{or}\quad \frac{a}{\sin A}=\frac{c}{\sin C}. \tag{3}$$

If the perpendicular were dropped on c, the same argument would give $a/\sin A=b/\sin B$. This with (2) gives

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C},$$

which is known as the ***law of sines***.

174. The Law of Cosines. In § 158, we saw that

(4) $a^2=b^2+c^2-2\,bx$, *if* $\angle\ A$ *is acute* (Fig. 133 *a*),

and

(5) $a^2=b^2+c^2+2\,bx$, *if* $\angle\ A$ *is obtuse* (Fig. 133 *b*).

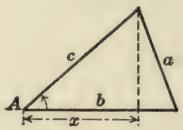


FIG. 133 *a*

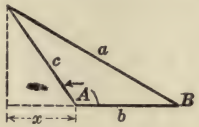


FIG. 133 *b*

But $x=c\cos A$, if A is acute and $x=-c\cos A$ if A is obtuse. Substituting these values of x in (4) and (5), respectively, we have, *for all values of* A, acute or obtuse,

(6) $$a^2=b^2+c^2-2\,bc\cos A,$$

or

(7) $$\cos A=\frac{b^2+c^2-a^2}{2bc}.$$

Either of these formulas is called the **law of cosines.** Similarly,

$$b^2=a^2+c^2-2\,ac\cos B, \quad \text{or} \quad \cos B=\frac{a^2+c^2-b^2}{2\,ac};$$

and

$$c^2=a^2+b^2-2\,ab\cos C, \quad \text{or} \quad \cos C=\frac{a^2+b^2-c^2}{2\,ab}.$$

175. The Use of Logarithms. Logarithms may be used to shorten computations involving multiplications, divisions, raising to powers, extraction of roots. They are often quite convenient but they are not essential. In the solution of triangles, the trigonometric functions are nearly always involved; hence, in the logarithmic solution, we need to find the logarithms of these trigonometric functions.

For example, if the hypotenuse c of a right triangle and an acute angle A are given, then the side a is

$$a=c\sin A.$$

To use logarithms to compute a we should then find $\log c$ and $\log (\sin A)$. Now *sin A is a number and log (sin A) means the logarithm of that number.* Thus, if $A = 26°$, then $\sin 26° = .4384$, and $\log (.4384) = 9.6418–10$. Hence

$$\log (\sin 26°) = 9.6418–10.$$

However, this double use of the table is not necessary, since these and similar results for other trigonometric functions are already tabulated in the table of trigonometric functions in the columns headed Log. Thus, in Table IX opposite 26°, under Sine, Log, we find .6418. The characteristic 9(–10) must be supplied as in the case of the logarithm of any number, by noticing the position of the decimal point in the *value* of the sine.

Examples. By use of the table, verify that

$\log \sin 15° = 9.4130$	$\log \tan 76° = 0.6032$
$\log \cos 26° = 9.9537$	$\log \tan 86° = 1.1554$
$\log \tan 26° = 9.6882$	$\log \sin 4° = 8.8436$

In these examples, as is usual in computations, the final –10 is omitted.

176. Checking the Solution. When a solution to a problem has been obtained, some test of its correctness should be applied. In general this may be done by obtaining the same solution by a different method and by examining the reasonableness of the solution.

In checking oblique triangles, the parts must be related as in I, II, III, § 171. A very great aid in testing a solution is the construction of the data to scale.

In applying logarithms to the checks, the logarithm of the computed part should be looked up in the table instead of using the logarithm from which the result was obtained. Logarithmic checks are not to be held to a closer agreement than the data call for. In solving triangles, except when three sides are given, there are three logarithms to be added

(subtracted). Hence the last place in the sum of these logarithms is in doubt by 1.5 (§§ 72, 73). When three sides are given, four logarithms may be combined. It is usually sufficient if the logarithmic checks agree within 2 in the last place.

EXAMPLE. In a right triangle with the following data, find the unknown parts and check the results.

Given the hypotenuse $c=27.3$, $\angle B=35°\ 15'$, to find $\angle A$, b, c.

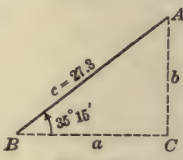


FIG. 134

Formulas:

(*a*) $A=90°-B=54°\ 45'$.

(*b*) $b=c \sin B$.

(*c*) $a=c \cos B$.

SOLUTION:

Using (*b*),

$\log c = 1.4362$

$\log \sin B = 9.7613$

$\log b = 1.1975$

$b = 15.8$

Using (*c*),

$\log c = 1.4362$

$\log \cos B = 9.9120$

$\log a = 1.3482$

$a = 22.3$

CHECK: $a \sin B = b \sin A$.

$\log a = 1.3483$

$\log \sin B = 9.7613$

1.1096

$\log b = 1.1987$

$\log \sin A = 9.9120$

1.1107

DISCUSSION. The given data are three-figured; hence the computed results are taken to the nearest three-figure approximations. Therefore in applying the check, we use the logarithms of 15.8 and 22.3 as found from the table, and *not* the logarithms of b and a from the solution. The logarithms in the check agree within 2 in the third place of mantissas, which is all that is required for three-figure data.

177. Case I. Given One Side and Two Angles. The third angle is easily found from the fact that $A+B+C=180°$. Each unknown side is found by the law of sines. For the sine of an *obtuse* angle, use the sine of its supplement, §§ 92, 93. Check by the law of sines, using in it that part not already used in the solution.

EXAMPLE. Given $a = 26.34$, $A = 80° 15'$, $C = 44° 27'$, to find b, c, and B.

FORMULAS:

(a) $B = 180° - (A + C) = 55° 18'$.

(b) $b = \dfrac{a \sin B}{\sin A}$. (See (1), § 173.)

(c) $c = \dfrac{a \sin C}{\sin A}$. (See (1), § 173.)

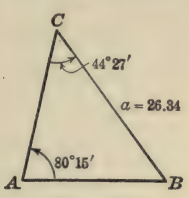


FIG. 135

SOLUTION:

From (b),

$\log a = 1.4206$
$\log \sin B = 9.9149$
sum $= 1.3355$
$\log \sin A = 9.9937$
$\log b = 1.3418$
$b = 21.97$

From (c),

$\log a = 1.4206$
$\log \sin C = 9.8453$
sum $= 1.2659$
$\log \sin A = 9.9937$
$\log c = 1.2722$
$c = 18.72$

CHECK: $b \sin C = c \sin B$. (See (1), § 173.)

$\log 21.97 = 1.3417$
$\log \sin C = 9.8453$
1.1870

$\log 18.72 = 1.2723$
$\log \sin B = 9.9149$
1.1872

The solution is thus valid as the logarithms here differ by only 2 in the last place of decimals.

EXERCISES

Solve and check each of the following oblique triangles, and find the area.

1. $a = 34.62$, $A = 62° 30'$, $C = 34° 10'$.

Ans. $b = 38.77$, $c = 21.92$, Area $= 376.9$

2. $b = 62.32$, $A = 37° 37'$, $B = 42° 15'$. *Ans.* $a = 56.57$, $c = 91.24$

3. $c = 123.4$, $B = 29° 18'$, $C = 134° 10'$. *Ans.* $a = 48.96$, $b = 84.19$

4. One side of a triangular field is 1234 feet long. The adjacent angles are 43° 43′ and 34° 34′. Find the perimeter of the field and its area in acres. *Ans.* Perimeter $= 2820$ ft. Area $= 7$ acres lacking 27 sq. ft.

5. A balloon is directly over a straight road. From two points $2\frac{1}{2}$ miles apart on this road, its elevation is observed to be 62° and 25°, respectively. Find the height of the balloon. There are two cases.

Ans. Height $= 8184$ feet when both points are on the same side of the balloon.

178. Case II. Given Two Sides and the Included Angle. The triangle can be divided into two right triangles, one of which contains two of the given parts, by drawing the altitude on one of the known sides.

Again, if the data are not complex, perhaps three-figured or less, the third side of the triangle may be found directly by use of the law of cosines. The other two angles may be found by the law of sines and the part of this formula not already used may be used as a check.

Both of these methods will be illustrated by examples.

EXAMPLE 1. Two sides of a triangle are 16 and 22 and their included angle is 53° 8′. Find the third side, the other two angles, and the area.

Given $b=16$, $c=22$, $A=53°\ 8'$, to find a, B, and C, and the area of ABC.

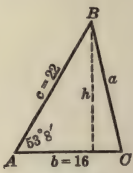


FIG. 136

FORMULAS:

(*a*) $a=\sqrt{b^2+c^2-2\,bc\cos A}$. (§ 174.)

(*b*) $\sin B=\dfrac{b\sin A}{a}$,

$\sin C=\dfrac{c\sin A}{a}$. (§ 173.)

(*c*) $\text{Area}=\dfrac{\text{base}\times\text{altitude}}{2}$.

SOLUTION: Using the law of cosines, we have

$$a=\sqrt{256+484-2(16)(22)(.6000)}=\sqrt{317.6}=17.8$$

From (*b*), we find

$$\sin B=\frac{16(.8000)}{17.8}=.7191;$$

hence $B=45°\ 59'$.

Likewise

$$\sin C=\frac{22(.8000)}{17.8}=.9888;$$

hence $C=81°\ 25'$.

Finally, by (*c*), we have

$$\text{Area}=\tfrac{1}{2}\text{ base}\times c\sin A=\tfrac{1}{2}\,16\times 22(.8000)=140.8$$

EXAMPLE 2. Given $b=41.02$, $c=45.49$, $A=62°\ 10'$, to find a, B, and C.

FORMULAS: Having drawn the altitude as shown in Fig. 137, we have

(a)	$x=c\cos A.$	(b)	$h=c\sin A.$
(c)	$y=b-x.$	(d)	$\tan C=h/y.$
(e)	$B=180-(A+C).$	(f)	$a=h\div\sin C.$

FIG. 137

SOLUTION:

From (a), we have

$$\log c=1.6580$$
$$\log\cos A=9.6692$$
$$\log x=1.3272$$
$$x=21.24$$

whence

$$y=19.78$$

From (b),

$$\log c=1.6580$$
$$\log\sin A=9.9466$$
$$\log h=1.6046$$
$$h=40.24$$

CHECK: $b\sin A=a\sin B.$

$$\log b=1.6130$$
$$\log\sin A=9.9466$$
$$1.5596$$

From (d),

$$\log h=1.6046$$
$$\log y=1.2963$$
$$\log\tan C=\ .3083$$
$$C=63°\ 49'$$

whence

$$B=54°\ 1'.$$

From (f),

$$\log h=1.6046$$
$$\log\sin C=9.9529$$
$$\log a=1.6517$$
$$a=44.84$$

$$\log a=1.6517$$
$$\log\sin B=9.9081$$
$$1.5598$$

EXERCISES

In each of the following exercises find the remaining parts, check the results, and find the area.

1. $a=56.70$, $b=45.20$, $C=47°\ 45'$.

Ans. $c=42.56$, $A=80°\ 26'$, $B=51°\ 49'$, Area $=948.7$

2. $b=576.2$, $c=431.6$, $A=73°\ 18'$.

3. $a=318.4$, $c=94.20$, $B=37°\ 37'$.

4. $c=14.61$, $a=6.234$, $B=111°\ 11'$.

5. Two forces of 18.2 lb. and 22.4 lb. act at a point on a body and at an angle of $57°\ 15'$ with each other. Construct the resulting parallelogram of forces and find each of the diagonals of this parallelogram.

6. B is 40 miles from A in direction N. 68° W., while C bears N. 17° E., 55.2 miles. What is the distance and direction of C from B?

Ans. 65.3 miles, N. 54° 39' E.

179. Case III. Three Sides Given. Find the angles by the law of cosines and check by the law of sines. Any small discrepancy between 180° and $A+B+C$ should be distributed among the angles so as to satisfy this requirement.

Whenever the square of one side is greater than the sum of the squares of the other two, thus making the numerator in (7), § 174, negative, ignore the minus sign while dividing, then *use for the result the supplement of the angle found from the table.*

For example, if $a=19$, $b=13$, $c=10$, we have

$$\cos A = \frac{b^2+c^2-a^2}{2\,bc}$$

$$= \frac{169+100-361}{260} = -(.3538)$$

This means that A is greater than 90°, and A is surely less than 180°. Therefore let us set $A'=180°-A$ and look up $\cos A'=+.3538$ We find from the tables $A'=76°\ 57'$; whence it follows that

$$A=180°-A'=103°\ 3'.$$

EXAMPLE. Given $a=42.1$, $b=54.3$, $c=50.7$, to find A, B, C, and the area.

FORMULAS:

(*a*) $\cos A = \dfrac{b^2+c^2-a^2}{2\,bc}$.

(*b*) $\cos B = \dfrac{a^2+c^2-b^2}{2\,ac}$.

(*c*) $C=180°-(A+B)$.

(*d*) Area $=\frac{1}{2}\,bc\sin A$.

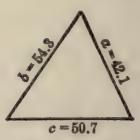


FIG. 138

SOLUTION: From (*a*), we have

$$\cos A = \frac{(54.3)^2+(50.7)^2-(42.1)^2}{2(54.3)(50.7)} = .6805$$

$$A=47°\ 7'.$$

From (*b*),

$$\cos B = \frac{a^2+c^2-b^2}{2\,ac}$$

$$= .3266$$

$$B = 70^\circ\ 56'.$$

$$C = 180^\circ - (A+B)$$

$$= 61^\circ\ 57'.$$

CHECK: $a \sin C = c \sin A$.

$\log a = 1.6243$	$\log c = 1.7050$
$\log \sin C = \underline{9.9457}$	$\log \sin A = \underline{9.8649}$
1.5700	1.5699

The check agrees in the third place of mantissas. This is sufficient, since the values of the lengths of the sides a, b, and c were given only to three significant figures.

Finally, from (*d*), the area of the given triangle may be calculated as follows.

$$\log b = 1.7348$$
$$\log c = 1.7050$$
$$\log \sin A = \underline{9.8649}$$
$$\overline{3.3047}$$
$$\log 2 = \underline{0.3010}$$
$$\log \text{Area} = \overline{3.0037}$$
$$\text{Area} = 1009.$$

EXERCISES

In each of the following exercises determine the remaining parts, check the results, and find the area.

1. $a = 14$, $b = 16$, $c = 10$.

Ans. $A = 60^\circ$, $B = 81^\circ\ 47'$, $C = 38^\circ\ 13'$.

2. $a = 122$, $b = 146$, $c = 112$.

3. $a = 32$, $b = 47$, $c = 62$.

[HINT. From a figure, it is seen that C is obtuse.]

4. $a = 132.4$, $b = 101.7$, $c = 84.50$

5. A farmer has a four-sided, irregular field. If the corners are denoted by A, B, C, D taken in order around the field and if $AB = 17$, $BC = 12$, $CD = 13$, $DA = 10$, and $BD = 16$, find the area and the angle at A.

180. The Law of Segments. In § 179, the triangle with three sides given was solved by use of the law of cosines. A second method for this case, better adapted to the use of logarithms, will now be given.

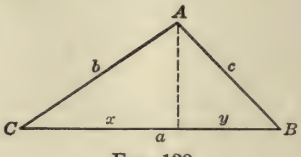


FIG. 139

Drop a perpendicular from A on a, dividing a into two segments x and y, as shown in Fig. 139. From the resulting right triangles, we have

$$b^2-x^2=c^2-y^2, \quad \text{or} \quad x^2-y^2=b^2-c^2, \tag{8}$$

or

$$(x+y)(x-y)=(b+c)(b-c). \tag{9}$$

From the figure, we see that

$$x+y=a. \tag{10}$$

Dividing the sides of equation (9) by the sides of (10), in order, we obtain the relation

$$x-y=\frac{(b+c)(b-c)}{a}. \tag{11}$$

Solving (10) and (11) by addition and subtraction, we find

$$x=\frac{a}{2}+\frac{(b+c)(b-c)}{2a}, \tag{12}$$

$$y=\frac{a}{2}-\frac{(b+c)(b-c)}{2a}. \tag{13}$$

But we know that

$$\cos C=\frac{x}{b}, \quad \text{and} \quad \cos B=\frac{y}{c}; \tag{14}$$

hence B and C can be found. Then

$$A=180^\circ-(A+B).$$

The solution may be checked by the law of sines.

EXAMPLE. Given $a=27$, $b=24$, $c=19$, to find A, B, C.

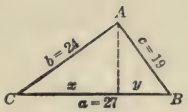


FIG. 140

FORMULAS:

(*a*) $x+y=a$.

(*b*) $x-y=\dfrac{(b+c)(b-c)}{a}$.

(Find x and y from (*a*) and (*b*) by addition and subtraction.)

(*d*) $\cos B=\dfrac{y}{c}$,

$\cos C=\dfrac{x}{b}$.

SOLUTION:

$$\begin{aligned} x+y &= 27 \\ x-y=\frac{(43)(5)}{27} &= 7.963 \\ \hline x &= 17.48 \\ y &= 9.52 \end{aligned}$$

$\cos B=\dfrac{9.52}{19}$

$\log 9.52=0.9786$

$\log 19=1.2788$

$\log \cos B=9.6998$

$B=59°\ 56'$

$A=76°\ 48'$.

$\cos C=\dfrac{17.48}{24}$

$\log 17.48=1.2424$

$\log 24=1.3802$

$\log \cos C=9.8622$

$C=43°\ 16'$.

CHECK: $b \sin C=c \sin B$.

$\log b=1.3802$

$\log \sin C=9.8360$

1.2162

$\log c=1.2788$

$\log \sin B=9.9372$

1.2160

EXERCISES

Use the law of segments to solve each of the following triangles. Check your answers.

1. $a=42.6$, $b=38.4$, $c=27.2$

2. $a=39.42$, $b=47.82$, $c=61.37$

3. Use the law of segments to check Exs. 3, 4, § 177.

4. Under what angle is a ship 850 feet long seen by a man who is standing on a wharf which is 600 feet from the bow of the ship and 950 feet from her stern?

181. Case IV. Given Two Sides and an Angle Opposite One of Them. Let the given parts be

$$a,\ b,\ A.$$

Construction of the given data to scale will usually indicate the possibilities of this case. See Case IV, § 171.

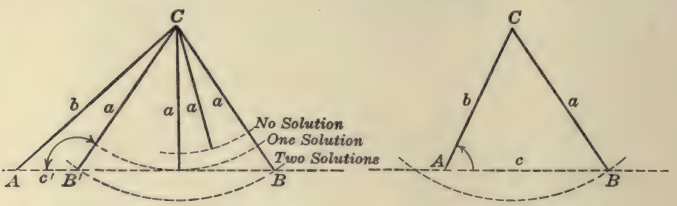


FIG. 141. — POSSIBLE CONSTRUCTIONS FOR THE AMBIGUOUS CASE

If $a<b$ and $\angle A$ acute, there are, Fig. 141,

two solutions if $a>$ the altitude on c, i.e. if $a>b \sin A$.
one solution if $a=$ the altitude on c, i.e. if $a=b \sin A$.
no solution if $a<$ the altitude on c, i.e. if $a<b \sin A$.

If $a>b$, there is one solution.

If $\angle A$ is obtuse, there is never more than one solution.

A second angle, $\angle B$ here, is always obtained from the relation $a \sin B = b \sin A$. As found from the table, $\angle B$ is acute. Then B', Fig. 141, is the supplement of B since $\triangle B'CB$ is isosceles.

This is often called the ***ambiguous case***.

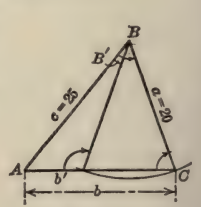


FIG. 142

EXAMPLE. Given $a=20$, $c=25$, $A=52°\ 40'$, to find b, B, C.

The construction shows that two triangles ABC, ABC' fulfill the conditions.

FORMULAS:

(a) $$\sin C = \frac{c \sin A}{a}.$$

(b) $$b = \frac{a \sin B}{\sin A}.$$

(c) $$C' = 180 - C.$$

(d) $$b' = \frac{a \sin B'}{\sin A}.$$

SOLUTION: From (a),

$\log c = 1.3979$
$\log \sin A = 9.9004$
$\text{sum} = 1.2983$
$\log a = 1.3010$
$\log \sin C = 9.9973$
$C = 83° 40'.$
$B = 43° 40'.$

From (b),

$\log a = 1.3010$
$\log \sin B = 9.8391$
$\text{sum} = 1.1401$
$\log \sin A = 9.9004$
$\log b = 1.2397$
$b = 17.4$

CHECK: $b \sin C = c \sin B.$

$\log 17.4 = 1.2405$
$\log \sin C = 9.9973$
1.2378

CHECK: $b' \sin C' = c \sin B'.$

$\log 13.0 = 1.1139$
$\log \sin C' = 9.9973$
1.1112

$C' = 180° - C = 96° 20'.$
$B' = 180° - (A + C) = 31°.$

From (d),

$\log a = 1.3010$
$\log \sin B' = 9.7118$
$\text{sum} = 1.0128$
$\log \sin A = 9.9004$
$\log b' = 1.1124$
$b' = 13.0$

$\log 25 = 1.3979$
$\log \sin B = 9.8391$
1.2370

$\log 25 = 1.3979$
$\log \sin B' = 9.7118$
1.1097

EXERCISES

In each of the following exercises find the remaining parts and check the results. Determine first in each case whether there is one solution, or two solutions, or none. Find the area.

1. $a = 50.8$, $b = 65.9$, $A = 41° 00'$. *Ans.* $B = 58° 21'$, $C = 80° 40'$, $c = 76.4$, $B' = 121° 40'$, $C' = 17° 20'$, $c' = 23.1$, Areas 1652, 499.

2. $b = 249.8$, $c = 409.1$, $B = 37° 38'$.

3. $b = 232$, $a = 107$, $B = 26° 36'$.

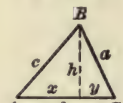
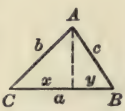
4. $a = 518.1$, $c = 694.7$, $C = 115° 42'$.

5. $b = 385.7$, $c = 491.2$, $B = 46° 15'$.

6. A man walking along a road due east sees a fort 10.2 miles away and bearing N. 57° E. If the guns of the fort have a range of 8 miles, how far may he go (a) before he comes within range of the guns, and (b) before he is again out of range?

7. A 50-ft. chord of a circle subtends an angle of 100° at the center. In the larger segment, a triangle with one side 57.8 ft. is to be inscribed. Find the length of the third side and the area of the triangle. How many solutions are possible?

182. Summary.

Case	Given	To Find	Formulas	Check
I. § 177	**One Side and Two Angles** as, $c=26.34$, $A=80°\ 15'$, $B=44°\ 17'$.	$C=$ $a=$ $b=$	$C=180°-(A+B)$. $a=\frac{c\sin A}{\sin C}$. $b=\frac{c\sin B}{\sin C}$.	$a\sin B=b\sin A$.
II. § 178	**Two Sides and the Included Angle** as, $b=41.02$, $c=45.49$, $A=62°\ 10'$.  Fig. 143	$a=$ $B=$ $C=$	1st Method: for simple data. $a=\sqrt{b^2+c^2-2bc\cos A}$. $\sin B=\frac{b\sin A}{a}$. $C=180°-(A+B)$. 2d Method: $x=c\cos A$. $h=c\sin A$. $y=b-x$. $\tan C=h/y$. $B=180-(A+C)$. $a=\frac{h}{\sin C}$.	$C\sin B=b\sin C$. $a\sin B=b\sin A$.
III. § 179 § 180	**Three Sides** as, $a=42.1$, $b=54.3$, $c=50.7$  Fig. 144	$A=$ $B=$ $C=$	1st Method: for simple data. $\cos A=\frac{b^2+c^2-a^2}{2bc}$. $\cos B=$etc.; $\cos C=$etc. 2d Method: Law of Segments. $x+y=a$. $x-y=\frac{(b+c)(b-c)}{a}$. $x=$(add) $y=$(subtract) $\cos B=\frac{y}{c}$; $\cos C=\frac{x}{b}$. $A=180°-(B+C)$.	
IV. § 181	**Two Sides and an Angle Opposite One of Them** as, $a=20$, $c=25$, $A=52°\ 40'$ and known as the **Ambiguous Case**	$C=$ $B=$ $b=$	$\sin C=\frac{c\sin A}{a}$. Thus $C=$an acute angle, and $C'=180°-C$. $B=180°-(A+C)$. $b=\frac{a\sin B}{\sin A}$.	$b\sin C=c\sin B$.

MISCELLANEOUS EXERCISES

1. P and Q are separated by a hill. To find QP a straight line AB is run through Q such that $AQ = 756$ ft., $QB = 562$ ft., $\angle QAP = 47° 29'$, and $\angle QBP = 57° 45'$. Find QP.

2. A man observes the angle of elevation of a hill to be $29° 18'$. After walking one mile directly toward the hill on a level plane, he finds that the angle of elevation is $36° 31'$. Find the height of the hill.

3. The parallel sides of a trapezoid are 36 and 52 and the non-parallel sides are 23 and 27. Find the angles and the diagonals.

4. A surveyor has the following data. $AB = 487$ feet, $BC = 179$ feet, $\angle CAB = 28°$. Show that something is wrong with the notes.

5. If a billiard table is a feet long and b feet wide, at what angle, I, must a ball be made to strike the long cushion in order that after striking each cushion in turn it will pass through its original position?

Ans. $\tan I = \frac{b}{a}$.

6. A trough is 10 ft. long and its cross-section is a semi-circle whose radius is 1 ft. Find the number of gallons of water it contains when the water measures 4 in. deep. When 6 in. deep. When 9 in. deep.

7. A road OA is $9\frac{5}{8}$ miles long and makes an angle of $31° 16'$ with a straight beach OX. From A two straight roads AB and AB' each 6 miles long run to the beach. Find the distance along the beach from O to the nearer of the points B and B'. (HARVARD.)

8. What angle does $y = 2x + 6$ make with the x-axis? What angle does $3y - 4x = 12$ make with the x-axis?

9. Plot the lines $2x + 3y = 12$ and $4x - 3y = 12$. Find their point of intersection and the angle each makes with the x-axis.

10. In the accompanying figure, O is the center of revolution of the crank pin of a steam engine, l is the connecting rod, P is the end of the slide connected with the piston head, and A is the angle of revolution POR. What is OP, when $R = 6''$, $l = 24''$, and $A = 30°$? Also if $A = 45°$? If $A = 135°$?

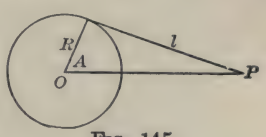


FIG. 145

Ans. $29''$; $27.9''$; $19.4''$

CHAPTER XIII

LAND SURVEYING

183. Introduction. To survey land means to measure distances and angles, to calculate from these measurements other distances and angles, to locate and mark the corners, to record these measurements and locations so that a ***plat*** or map can be made, and to find the area of the tract or parcel of land.

Sometimes it is desirable to *resurvey* an original tract. The surveyor then seeks to find the original corners, to retrace the boundaries, and to replace any landmarks, such as posts, monuments, etc., that have decayed or have been destroyed. In addition to this he is often called upon to subdivide tracts of land into smaller parcels, as in city subdivisions.

184. Instruments. For measuring distances surveyors usually use a ***steel tape*** 100 feet long, or a ***chain*** which is 66 feet = 4 rods long and which contains 100 links. For these and other ***units of measurement***, see § 185. The surveyor also uses ***pins*** to mark the successive applications of the tape or chain; ***range poles*** for sighting and ranging out directions; and a graduated rod or pole called a ***leveling-rod*** provided with a sliding target by which he can measure elevations and depressions, as in establishing the grade for a ditch.

Almost all of the old land surveys of this country were made with the ***compass*** as the angle-measuring instrument.

However, it has now been supplanted to a great extent by the transit.

The *transit* (Fig. 146) is the instrument now most used by surveyors and engineers for measuring horizontal angles, and, with certain attachments, for measuring vertical angles

Fig. 146

and distance, and for leveling. It is mounted on a tripod and consists essentially of the following three parts.

(*a*) the *telescope* to determine direction. To the telescope is attached a level and a vertical graduated circle to determine angles of elevation and depression.

(*b*) the ***alidade***, a horizontal circular piece provided with vertical supports to carry the telescope, with a graduated circle, and with precise levels so that this circle can be brought exactly into the horizontal plane. Angles in this plane may be read on this circle as the angular turn of the telescope.

(*c*) the ***magnetic compass.***

185. Surveyor's Measures. City lots are usually surveyed in terms of feet and square feet, with decimal parts of these units. However, in measuring larger pieces of land the following measures are used.

1 ***rod***	$= 16\frac{1}{2}$ feet.
	= 1 ***perch*** = 1 ***pole*** (older plan).
1 ***chain*** (ch.)	= 4 rods = 66 feet.
	= 100 ***links*** (li.).
1 ***link***	= 7.92 inches.
1 ***Acre*** (A.)	= 160 square rods (sq. rd.).
	= 10 square chains (sq. ch.).
1 ***section***	= 640 acres.

In some states ***Spanish-American units*** are used as follows:

1 ***Vara*** (vä′-ra)	$= 33\frac{1}{3}$ inches in Texas,
	= 33 inches in California,
	= 32.9927 inches in Mexico.
1 ***labor*** (lä-bōr′)	= a square whose side is 1000 varas,
	= 1,000,000 square varas,
	= 173 acres (approximately),
	= a quarter section (roughly).
1 ***league***	= a square whose side is 5000 varas,
	= 25 labors,
	= 6.8 square miles (approximately).
1 (Spanish) ***Acre***	= 5645 square varas (in Texas).

Since 100 links = 1 chain, the older plan of writing both chains and links in a survey is discarded and distances are written in terms of decimals of a chain. For example, 11 chains 24 links is written 11.24 ch. Similarly, the area is written in decimals of an acre, as 2 acres 4 square chains = 2.4 A., 2 A. 47 sq. rd. = 2.294 A.

186. General Procedure as to Corners, Offsets, Obstacles, Measuring Slopes, etc. When a corner has been determined in a survey, and agreed upon by all parties concerned, it should be marked with stones, iron or concrete posts, etc., in such a way as to perpetuate its location as long as possible.

In measuring distances two men, a ***leader*** and a ***follower***, carry the tape or chain. The leader carries ten pins, and at each measurement he "sticks" a pin to mark it. On moving forward the follower picks up this pin. The process is repeated and tallied, both leader and follower keeping their own tallies.

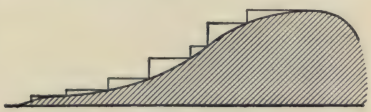


FIG. 147. — MEASURING ON A SLOPE

Since areas are computed as if the land were flat, it is necessary in measuring on slopes to keep the chain level (Fig. 147).

It sometimes happens, on account of fences, brush, ponds, or other obstructions, that a measurement cannot be made

FIG. 148

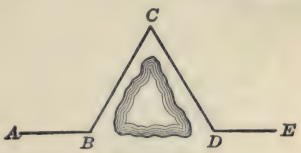


FIG. 149

on the desired line. The surveyor then measures a convenient portion of a parallel to the required line at a certain perpendicular distance from it, as $CD = BE$, Fig. 148.

An obstacle sometimes may be passed and the general direction continued beyond it by measuring an equilateral triangle about the obstacle as in Fig. 149.

187. Computing the Area. In §§ 18 and 42, methods for finding the area of irregular shaped tracts have been discussed. These methods will now be discussed more fully, making use of the latitudes and longitudes of the corners of the survey. These may be looked upon as the *coördinates* of the vertices of the survey and follow from the lengths and bearings of the courses.

188. Definitions. If the y-axis be a meridian and the x-axis an east-west line, then *the ordinate of a point is called its* ***latitude***, *its abscissa is its* ***longitude***.

Let AB be any course, Fig. 150. Draw a meridian through A and perpendiculars BB', AA' to it and the y-axis. Then $PA = OA'$ *is the latitude of* A, OB' *is the latitude of* B, *and thus* $A'B' =$ ***the latitude-difference*** *of the course* AB. Similarly AA', BB' *are the longitudes of* A *and* B, respectively; and $KB =$ ***the longitude-difference*** *of the course* AB.

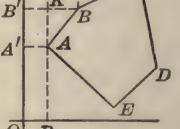


FIG. 150

Latitude-difference north, i.e. a northing, is taken to be *positive*, a *southing* is taken to be *negative*. Similarly, *longitude-difference east, i.e.* an *easting*, is *positive*, a *westing* is *negative*.

The ***double longitude of a course*** is the sum of the longitudes of its ends. Thus, in Fig. 150, the double longitude of AB is $AA' + BB'$.

189. Area when Lengths and Bearings of Courses are Given. By projecting the courses of a tract as $ABCDEA$, Fig. 151, on a meridian, we see that to every course there

corresponds a trapezoid. *Then to find the area of $ABCDEA$ we add the trapezoids $CC'DD'$ and $DD'EE'$ corresponding to courses which bear south and from this sum subtract the sum of the trapezoids corresponding to courses which bear north.* The areas of the one set are called ***south areas***, the others ***north areas***.

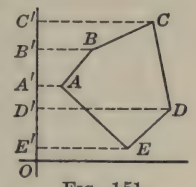


FIG. 151

Again, *the double area of any of these trapezoids* as $AA'BB'$ is numerically equal to the product of the double longitude of the corresponding course times its latitude-difference. Thus the double area of $AA'BB' = (AA' + BB')\,A'B'$. We have, therefore, the following rule.

RULE TO FIND THE DOUBLE AREA OF A TRACT. *Multiply the double longitude of each course by its latitude-difference, placing the product in a column of north areas when the bearing is north, and in a column of south areas when the bearing is south. Add up the two columns and take the difference of the results.*

190. Latitude-differences, Longitude-differences, and Double Longitudes. From Fig. 152, $KA = AB \cos KAB$; $KB = AB \sin KAB$, *i.e.* for any course, we have

Latitude-difference = length of the course times cosine of its bearing,

Longitude-difference = length of the course times sine of its bearing.

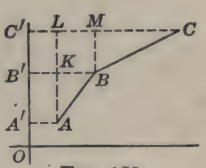


FIG. 152

From Fig. 152, let AB and BC be any two successive courses. Then the double longitude of $BC = BB' + CC' = (BB' + AA') + KB + MC$.

That is, ***the double longitude of any course*** *equals the double longitude of the preceding course, plus the longitude-difference of that course, plus the longitude-difference of the given course, eastings counted positive and westings negative.*

If the reference meridian is taken *one chain* west of the most westerly corner and the survey begins at that corner, then for the first course

$$\begin{aligned}\textit{the double longitude of } AB &= AA'+BB',\\ &= AA'+B'K+KB,\\ &= 1+1+\text{longitude-difference of } AB.\end{aligned}$$

Thus, double longitude of the first course equals 2 plus its longitude-difference.

If the reference meridian passes through the most westerly corner, then the double longitude of the first course is merely its longitude-difference.

191. Balancing the Survey. For a closed survey we should have

$$\textit{northings} = \textit{southings},$$
$$\textit{eastings} = \textit{westings}.$$

If these are not equal, the difference is called the ***error of closure.*** This error, if not too large, may be distributed among the sides in proportion to their respective lengths so that the survey will close. This is called ***balancing the survey.***

This and other details of computation and its arrangement will be illustrated in the example, § 192.

192. Illustrative Example. The field notes for a tract of land are given in the following table. Plot this data to scale, balance the survey for error of closure, and find the area.

Course	*AB*	*BC*	*CD*	*DE*	*EA*
Bearing	N. 40° E.	N. 60° E.	S. 20° E.	S. 60° W.	N. 38¼° W.
Distance	7.18 ch.	9.20 ch.	15.22 ch.	11.42 ch.	12.72 ch.

Plot the data to scale and take the reference meridian one chain west of the most westerly corner, *i.e.* $AA'=1$ **ch.** (Fig. 153).

Latitude-differences and longitude-differences must next be computed and balanced. For the first course AB, we have,

$$\text{lat.-diff.} = (7.18)\cos 40°$$
$$= (7.18)(.7660) = 5.50$$
$$\text{long.-diff.} = (7.18)\sin 40°$$
$$= (7.18)(.6428) = 4.61$$

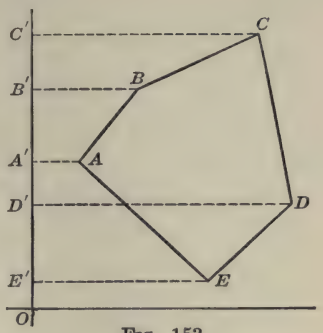


FIG. 153

These and the results for the other courses may be tabulated as follows.

COURSE	BEARING	DISTANCE (CHAINS)	LAT.-DIFF.		LONG.-DIFF.	
			N.	S.	E.	W.
AB	N. 40° E.	7.18	5.50		4.61	
BC	N. 60° E.	9.20	4.60		7.97	
CD	S. 20° E.	15.22		14.30	5.21	
DE	S. 60° W.	11.42		5.71		9.89
EA	N. 38¼° W.	12.72	9.99			7.87
		55.74	20.09	20.01	17.79	17.76
			20.01		17.76	
		Error	.08		.03	

The error in latitude-differences is .08, in longitude-differences .03. The survey will be *balanced* or *closed* by distributing these errors among the several courses in proportion to their lengths. The perimeter of the tract is 55.74 ch.; hence

$$\text{error in latitudes for 1 chain} = \frac{.08}{55.74} = .0014,$$

$$\text{error in longitudes for 1 chain} = \frac{.03}{55.74} = .0005$$

Therefore, for $AB = 7.18$ chains, the error in the latitude is assumed to be $(7.18)(.0014) = .01$, and the error in the longitude is assumed to be $(7.18)(.0005) = .00$

Similar results may be computed and tabulated for the other courses *BC*, *CD*, *DE*, and *EA*.

The northings are too large and each must be diminished by the corresponding corrections; the southings are too small and each must be increased; similarly for the eastings and westings. On making these corrections the work will appear as in the table on page 227.

To find the area, double longitudes are used. These are computed as follows from the rule in § 190.

long.-diff. of 1st course =	4.61
add	2.
double longitude of 1st course =	6.61
long.-diff. of 1st course =	4.61
long.-diff. of 2d course =	7.97
double longitude of 2d course =	19.19
long.-diff. of 2d course =	7.97
long.-diff. of 3d course =	5.20
double longitude of 3d course =	32.36
long.-diff. of 3d course =	5.20
long.-diff. of 4th course =	−9.90
double longitude of 4th course =	27.66
long.-diff. of 4th course =	−9.90
long.-diff. of 5th course =	−7.88
double longitude of 5th course =	9.88

CHECK: From a figure it is easily seen that the double longitude of the last course is *numerically* equal to its longitude-difference plus 2. This furnishes a check on the accuracy of the work. If the reference meridian is made to pass through the most westerly corner, then the *double longitude of the last course is equal numerically to its longitude-difference.*

These values of the double longitudes may now be inserted in the table and the double areas computed by the rule of § 189. The work then takes the following final form.

Course	Bearing	Dist. (Ch.)	Lat.-Diff.		Long.-Diff.		Balanced				Double Longitude	Double Area	
			North-ing	South-ing	Easting	Westing	N.	S.	E.	W.		North	South
AB	N. 40° E.	7.18	5.50		4.61		5.49		4.61		6.61	36.2889	
BC	N. 60° E.	9.20	4.60		7.97		4.59		7.97		19.19	88.0821	
CD	S. 20° E.	15.22		14.30	5.21			14.32	5.20		32.36		463.3952
DE	S. 60° W.	11.42		5.71		9.89		5.73		9.90	27.66		158.4918
EA	N. $38\frac{1}{4}$° W.	12.72	9.99			7.87	9.97			7.88	9.88	98.5036	
		55.74	20.09	20.01	17.79	17.76	20.05	20.05	17.78	17.78		222.8746	621.8870
													222.8746
													2)399.0124

Area in sq. ch. = 199.5062
Area in acres = 19.95

EXERCISES

Plot the following described plats of land to scale, balance the survey for error of closure, and find the area in acres.

1.

Course	*AB*	*BC*	*CD*	*DE*	*EA*
Bearing	N. 21° E.	N. 65° E.	S. 33° E.	S. 65½° W.	N. 48° W.
Distance	57.16 ch.	7.68 ch.	52.20 ch.	50.96 ch.	13.40 ch.

Ans. 187.13 A.

2.

Course	*AB*	*BC*	*CD*	*DE*	*EA*
Bearing	S. 23° E.	S. 80° E.	N. 38° E.	N. 42° W.	S. 60¼° W.
Distance	48 ch.	52 ch.	46 ch.	62 ch.	64 ch.

Ans. 666.36 A.

3.

Course	*AB*	*BC*	*CD*	*DE*	*EA*
Bearing	N. 41° E.	S. 70° E.	S. 29° E.	West	N. 49¼° W.
Distance	60 ch.	40 ch.	72 ch.	75 ch.	48 ch.

Ans. 32.89 A.

4.

Course	*AB*	*BC*	*CD*	*DE*	*EA*
Bearing	N. 47° E.	East	S. 32° E.	South	N. 77° 53′ W.
Distance	964 ch.	822 ch.	644 ch.	523 ch.	1961 ch.

Ans. 235.79 A.

5. In the example of § 192, find the length and bearing of *AC* and of *AD*.

6. Assuming that the measurements in the following description were all correctly made, find the length of the last course and compute the area.

Beginning at an iron pin in the center of the township road and at the northeast corner of said tract, thence N. 89° W., 140.8 rods to a stone set in the center of the road, thence S. 29° 10′ W., 65.7 rods to a large boulder, thence S. 21° 18′ E., 42.3 rods to an 8″ × 8″ concrete post set in the ground on the south bank of the creek, thence S. 65°

36′ E., 38.6 rods to an iron pin in the center of the Salem-Brookville turnpike, thence N. 59° 14′ E., 150.6 rods to an iron pin at the cross-roads, thence N. 2° W. to the place of beginning.

7. Balance the survey and compute its area from the following field notes.

N. 27° E., 366.0 feet; N. 19° W., 354.0 feet; S. 69° W., 655.5 feet; S. 19° E., 275.3 feet; S. 71° E., 486.0 feet. *Ans.* 6.732 A.

193. Locating by Metes and Bounds. The system of describing and locating lands by metes and bounds is used extensively over the world to-day and, quite naturally, was introduced into this country by the early settlers from Europe. To locate a desired area or region by landmarks, some permanent, or well-defined, point of beginning is first established. The boundary lines are then described in direction and distance with reference to this point of beginning and with reference to more or less permanent natural objects such as trees, streams, well-established highways, as well as concrete monuments, stones, iron posts, etc., placed for the purpose. The directions are by magnetic compass and the distances are in terms of the surveyor's units. However, the fact that landmarks decay and change and, that *the magnetic direction of the same line changes from year to year*, gives a transitory character to the platting of land in this way.

194. General Plan of Government Surveys. In 1785 practically all of the territory north and west of the Ohio River had been ceded to the United States Government by the withdrawal of state claims. A law passed by Congress May 20, 1785, provided that

"The surveyors . . . shall proceed to divide the said territory into townships six miles square, by lines running due north and south and others crossing these at right angles, as near as may be."

Owing to the convergence of the meridians this, of course, was a mathematical impossibility. The phrase "as near as may be," however, has been broadly interpreted. According to the provisions of this act and the acts of May 18, 1796, May 10, 1800, and Feb. 11, 1805, and to the rules of the commissioners of the general land office, a complete system exists, whose main features are as follows.

First. Land is to be surveyed or marked out in divisions in the form of squares or rectangles. This has been done in the western states, including, roughly, all of the states of the Union except the original thirteen Colonies.

Second. To mark out the largest squares, north and south lines, called ***meridians***, are first laid off 24 miles apart and marked with cornerstones, trees, or other permanent objects. East and west lines, called ***base lines***, are then run at right angles to these meridians at intervals of 24 miles. This would divide the land up into 24-mile squares were it not for the convergence of the meridians towards the poles of the earth. This may be noticed on maps in geographies and in Fig. 154. Each 24-mile tract is then divided into sixteen nearly equal squares called ***townships*** by running north-south and east-west lines through the quarter points of the sides of the large tract.

Certain meridians called ***principal meridians*** are located with great care. They govern the surveys of lands lying along them for considerable distances both toward the east and toward the west. The vertical rows of townships parallel to the principal meridian are called ***ranges***. The first row on the east is called range No. 1 and is written $R1E$; the second row east is No. 2, and is written $R2E$, etc.

Similarly, ***standard base lines*** are located. The tiers of townships parallel to this base line are numbered north and south of it. A township in the first tier north of the base

line is township No. 1 and is written *T1N*; one in the second tier south is No. 2 and is written *T2S*, etc.

A township is thus identified by its number and its range from some standard base line and principal meridian.

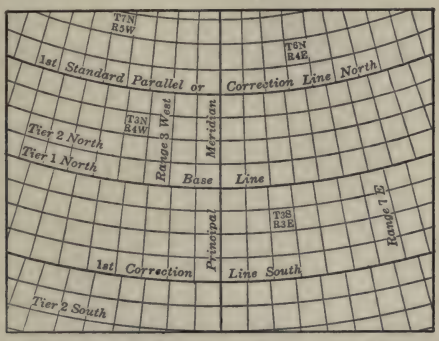


FIG. 154. — BASE LINE AND PRINCIPAL MERIDIAN

Point out the following townships in Fig. 154. *T1N, R2W*; *T3N, R4W*; *T2S, R7W*; *T3S, R3E*.

Third. Townships are required to be divided into smaller squares called ***sections*** one mile each way and containing 640 acres, "as near as may be." These sections are numbered as in Fig. 155, except the "first seven ranges" laid

6	5	4	3	2	1
7	8	9	10	11	12
18	17	16	15	14	13
19	20	21	22	23	24
30	29	28	27	26	25
31	32	33	34	35	36

FIG. 155

36	30	24	18	12	6
35	29	23	17	11	5
34	28	22	16	10	4
33	27	21	15	9	3
32	26	20	14	8	2
31	25	19	13	7	1

FIG. 156

31	32	33	34	35	36
30	29	28	27	26	25
19	20	21	22	23	24
18	17	16	15	14	13
7	8	9	10	11	12
6	5	4	3	2	1

FIG. 157

off by Hutchins west of the Pennsylvania line. (See § 196.) These are numbered as in Fig. 156. In western Canada the sections are numbered as in Fig. 157.

195. Standard Parallels or Correction Lines. The eastern and western boundaries of townships are, as nearly as may be, true meridians. Hence, when they are extended northward through several tiers of townships their convergence becomes considerable. At latitude 40° the convergence is about 6.7 feet per mile or over 40 feet to each township. To prevent this diminution in the size of townships to the north of the base line, standard parallels are run, along which six-mile measurements are made for a new set of townships. These lines are called ***correction lines*** for obvious reasons.

The standard parallel or correction lines have been run at varying distances, the present distance being 24 miles. Public roads are usually built on the section lines, and wherever a north and south road crosses a correction line there is a " jog " in the road as shown on Fig. 154.

196. Principal Meridians and Base Lines of the United States. The first public land survey in the United States was started in 1786 by Thomas Hutchins, Geographer of the United States, who, with thirteen assistants, laid off a line from the southwest corner of Pennsylvania due north to a point on the north bank of the Ohio River. The extension of this line north of the Ohio is known as Ellicott's Line. From this point on the north bank of the Ohio River, Hutchins started a line westward as a Base Line. According to Congressional direction, he was to lay off meridians at intervals of six miles along this east-west " Geographer's line " and also parallels to this east-west line, every six miles. Each of these six-mile squares was to be divided into thirty-six square miles and each of these into " quarters." The large squares were called " townships " after the New England word " town." Hutchins and his assistants laid off only forty-two miles of the " Geographer's line," making

seven ranges of townships west of the Pennsylvania state boundary line. They were frightened away by the Indians.

The Principal Meridians, § 194, are run from some initial point selected with great care and are located in latitude and longitude by astronomical means. More than thirty-two of these principal meridians have been surveyed at irregular intervals and of varying lengths. Some of them are numbered, while others are named. The first principal meridian (written, First P.M.) is the boundary between Indiana and Ohio; the second is west of the center of Indiana, extending the entire length of the state; the third extends the entire length of the state of Illinois and through the center; the Tallahassee principal meridian, only twenty-three miles long, runs directly through that city. Other principal meridians are the Black Hills, Indian, Louisiana, Mount Diablo, San Bernardino, etc.

A Base Line is run at right angles to the principal meridian at some point selected to begin the survey. Base lines are true geographical parallels, and as with the principal meridian, they are laid off with great care.

The fourth P.M. in western Illinois and Wisconsin has two base lines, one at its southern extremity and the other on the boundary between Illinois and Wisconsin.

The fifth P.M. and its base line carries the largest area embraced in any one system. This meridian extends from the mouth of the Arkansas River northward through Arkansas, Missouri, and Iowa. Its base line, known sometimes as the Little Rock base line, passes a few miles south of that city in Arkansas. From this meridian and its base line all of Arkansas, Missouri, Iowa, North Dakota, and the major portions of Minnesota and South Dakota have been surveyed. This constitutes an area considerably larger than that of France and Great Britain and Ireland com-

bined. The most northern tier of townships from this base line is near the forty-ninth parallel, the boundary between the United States and Canada, and is numbered 163. The most northern township lies more than a thousand miles north of the base from which it was surveyed. There are nineteen tiers south of this Little Rock base line, so that the extreme length of this area is about 1122 miles. The most eastern range from this fifth P.M. is number 17 and its most western 104, making an east and west extent of 726 miles.

CHAPTER XIV

SIMPLE MACHINES

197. Introduction. Due to wider and more intense competition, both with man and nature, the modern farmer cannot expend his strength in developing mere mechanical power which may be created by a horse or a windmill or an engine. He must devote a larger and larger share of his time and energy to mental activity. The result of this tendency is to increase the number and the complexity of his machines. Thus a knowledge of mechanical principles has become a fundamental necessity in order that he may properly handle and care for machinery.

198. Motion. Force. Moment of a Force. *Motion* is change of position. That which produces motion or changes it (or tends to do so) is called ***force.***

Force and motion are transmitted and modified primarily by the use of machines. Simple examples of machines are a crowbar used to raise a heavy weight, a pair of pliers in use, an ordinary lift pump, a pulley, etc.

If a force P (Fig. 158) is applied to the end of a lever working over the fulcrum F, it tends to lift the weight W, thereby producing a tendency to rotate about F as an axis. This tendency to rotate is called the ***moment of the force.*** The perpendicular distance from the axis of rotation to the line of action of the force is called the ***moment arm.*** *The moment of the force is measured by the force times the moment arm:*

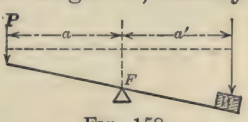


FIG. 158

Moment of a Force = Force × Moment Arm.

From Fig. 158,

Moment of P about the axis at $F = P \times a$,
Moment of W about the axis at $F = W \times a'$;

and for equilibrium

(1) $$P \times a = W \times a'.$$

199. Levers. In every simple machine two forces are involved, viz. the *resistance*, or the force to be overcome, and the *effort*, or the force necessary to overcome the resistance. The relation between these two forces depends upon the nature and the dimensions of the machine.

With levers, resistance and effort are inversely proportional to their distances from the fulcrum, i.e.

$$W : W' = l' : l, \quad \text{or} \quad Wl = W'l', \quad \textit{i.e.}$$

Resistance (W) *times its moment arm* (l)
= *effort* (W') *times its moment arm* (l').

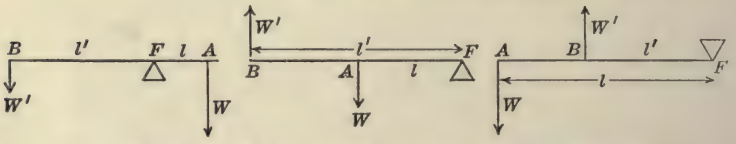


FIG. 159. — LEVERS, SHOWING FULCRUM AND MOMENT ARM

The beam balance, the common steelyard, scissors, pincers, the crowbar, etc. are familiar examples of levers.

200. Wheel and Axle. *With the wheel and axle, resistance and effort are inversely proportional to the radii of the cylinder and wheel respectively*, i.e.

$$W : W' = r' : r, \quad \text{or} \quad Wr = W'r'.$$

The windlass and the capstan are familiar examples of the wheel and axle. Wheels on separate axles, connected

by belts or by teeth in the circumference of the wheels, are modifications of the wheel and axle.

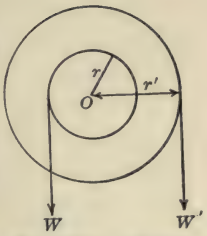


FIG. 160. — WHEEL AND AXLE

201. A General Principle. Work. If, in any simple machine, a body is made to move a distance d directly against a resisting force W, by an effort W' that moves a distance d' in the line of action of W', the product $W \cdot d$ is equal to the product $W' \cdot d'$. This product is called the ***work done.***

202. The Inclined Plane. *With the inclined plane, weight and effort are directly proportional to the length and the height of the incline,* i.e.

(2) $$W : W' = l : h, \text{ or } Wh = W'l.$$

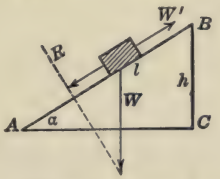


FIG. 161. — INCLINED PLANE

203. The Screw. *The screw is a modified form of the inclined plane.* The effort is applied at the end of a lever and the length of the inclined plane is the distance traversed by the effort in one revolution, *i.e.* $2\pi l$, where l is the lever arm. The height of the incline is the distance d between two successive threads of the screw. This distance, d, is called the ***pitch of the screw.***

Here, *the weight and effort are directly proportional to the distance traversed by the effort in one revolution, and the pitch of the screw,* i.e.

(3) $$W : W' = 2\pi l : d, \quad \text{or} \quad Wd = 2\pi W'l.$$

EXERCISES

1. What effort is required to lift a weight of 800 lb. by a lever whose weight arm is 6 in. and whose effort arm is 2 ft. 6 in.?

2. A pump handle is 3 ft. 8 in. long and works on a pivot 4 in. from the end attached to the pump rod. What force is applied to the pump rod when the end of the handle is pushed down with a force of 10 lb.?

3. A man wishes to lift a stone weighing 1000 lb. by means of a crowbar 6 ft. long. If he places the fulcrum 10 in. from the end of the bar, what force must be exerted at the end of the bar to raise the stone? In what direction must this force be applied? If the force is applied at a different angle, does it take more or less force? For example, if the bar stands at an angle of 45° when the force is applied, in what direction should the force be applied?

4. What force is required to lift a load of 275 lb. on a wheelbarrow if the load is placed 18 in. back of the axle of the wheel and the ends of the handles are 5 ft. from the axle?

5. The crank to the windlass of a well is 18 in. long and the cylinder upon which the rope is wound is 6 in. in diameter. How much force is necessary to lift a bucket of water weighing 48 lb., neglecting friction? How much if the friction is 20% of the force applied?

6. Two men working at a capstan walk in a circle 8 ft. in diameter and each exerts a force of 60 lb. With each complete turn, 2 ft. of rope is pulled in. What is the pull along the rope, neglecting friction?

7. A horse walking in a circle 15 ft. in diameter moves a house by means of a capstan 18 in. in diameter. The horse exerts a pull of 1200 lb. What is the resistance of the house if only $\frac{2}{3}$ of the force exerted by the horse is available?

8. A stone weighing 756 lb. is lifted into place in a wall by means of a cable wound about an axle of 5 in. radius. To this axle is rigidly attached a toothed wheel of 15 in. radius, and this toothed wheel is driven by another toothed wheel of 4 in. radius. This last wheel is turned by a crank 18 in. long. How much force must be applied to the crank, neglecting friction? How much, allowing 20% friction?

9. If a barrel of cider weighs 200 lb., what force is required to roll it up an incline 8 ft. long into a wagon 3 ft. high? How long would the incline have to be if a man could push only 40 lb.?

10. A stone weighs 850 lb. What force is required to pull it on rollers up an incline 12 ft. long and place it upon a wagon 30 in. high?

11. What force is required to pull a loaded wagon weighing 3800 lb. up a slope 30 ft. high and 105 ft. long, neglecting friction?

12. Ice is pulled up an incline 50 ft. long 20 ft. high into an ice house. What force, neglecting friction, is necessary to pull up a 200-lb. block?

13. A house that is being raised by jackscrews exerts a pressure of three tons on one of them. If the pitch of the screw is $\frac{3}{8}$ in. and it is operated by a lever 2 ft. long, what force must be applied at the end of the lever?

14. If the pitch of a jackscrew is $\frac{3}{4}$ in. and the lever arm 30 in., what weight may be lifted by it with a force of 100 lb.?

15. How far will a force of 180 lb. which moves 4 ft. lift a weight of 960 lb., if one third of the applied force is consumed in overcoming friction? *Ans.* 6 in.

16. What force, moving 6 ft. 8 in., will lift 512 lb. a distance of 1 ft. 3 in., if one third of the applied force is consumed in overcoming friction? *Ans.* 144 lb.

17. CD is a crowbar $6\frac{1}{2}$ ft. long, and is supported at F, 6 in. from C. How many pounds of force must be exerted by a man pressing down at D to raise a stone weighing 1800 lb. at A? To get the maximum of the force exerted at D, in what direction should it be applied?

18. Suppose, in Ex. 17, that the crowbar stands at an angle of 30° with the horizontal. A man weighing 150 lb. sits at D on the end of the bar. What is the force communicated to the bar at D? How much will it lift at C?

19. In pulling a nail from a board, as shown in the accompanying figure, find the resistance of the nail in starting if

$$P = 40 \text{ lb.}, \; D = 10'', \; d = 1\tfrac{1}{2}''.$$

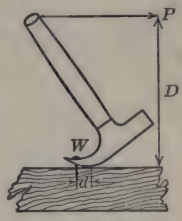


FIG. 162

20. If a gate weighs 50 pounds and is 10′ long, what is the pull on the upper hinge if it is 3′ above the lower hinge?

[HINT. The center of mass is in the center of the gate. The system of forces then acts as a simple teeter board with the fulcrum on the lower hinge.] *Ans.* $83\frac{1}{3}$ lb.

204. The Pulley. The pulley consists of a ***grooved wheel*** capable of revolving about an axis, fixed into a framework called the ***block***. When the axis of the pulley is fixed, the pulley is called a ***fixed pulley***. It has no mechanical advantage,* its use being merely to change the direction of the force.

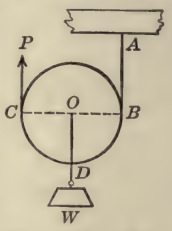


FIG. 163. — SIMPLE MOVABLE PULLEY

If the pulley can ascend or descend, it is called a ***movable pulley*** and a mechanical advantage may be gained. In Fig. 163, the weight W is supported by the force P applied to the cord $ABDP$ fixed at A and passing under the pulley. The tension in the cord being everywhere the same, we have

(4) $P \times$ lever arm $CB = W \times$ lever arm OB.

But $CB = 2\,OB$. Therefore

(5) $$P = \frac{W}{2}, \quad \text{or} \quad W = 2\,P;$$

and the mechanical advantage is 2, since $W/P = 2$.

205. System of Movable Pulleys. In Fig. 164 is shown a system of three movable pulleys each hanging from a fixed beam by a separate string.

The tension at A in the cord under the lowest pulley is $= W/2$. The tension at B in the cord under the second pulley is one half of the first tension, *i.e.* at A. Thus the tension at B is

$$\frac{W}{2} \cdot \frac{1}{2} = \frac{W}{2^2}.$$

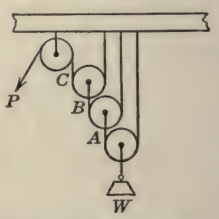


FIG. 164. — SYSTEM OF PULLEYS

* ***Mechanical advantage*** is the ratio of the resisting force W to the effort force P, *i.e.*

$$\text{mechanical advantage} = \frac{W}{P} = \frac{\text{distance } P \text{ moves}}{\text{distance } W \text{ moves}}.$$

The tension in the cord at C is one half that of the previous one, etc. to the nth pulley for which the tension in the cord supporting it is $W/2^n$. Thus, for n pulleys arranged as in Fig. 164, we have $P = W/2^n$.

206. Block and Tackle — Cord around All Pulleys. In this system there are two blocks, A and B, Fig. 165, the upper of which is fixed and the lower movable, and each containing a number of pulleys or ***sheaves***, each pulley movable around the axis of the block in which it lies. A single cord is attached to one of the blocks and passes alternately round the pulleys in the two blocks, the portions of the cord between successive pulleys being parallel. In the tackle shown in Fig. 165, there are three pulleys in each block.

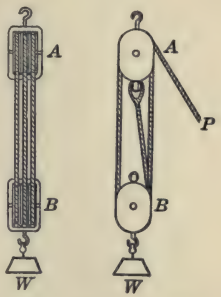


FIG. 165. — BLOCK AND TACKLE

The mechanical advantage of the block and tackle varies directly as the number of cords supporting the weight.

For, as the cord passes round all the pulleys, its tension is the same throughout and equal to P. Then, if n be the number of cords at the lower block, nP will be the resultant total upward tension of the cords at the lower block, which must equal W. Therefore

$$nP = W, \quad \text{or} \quad P = \frac{W}{n}. \tag{6}$$

In the system of Fig. 165, there are six cords supporting the weight at the lower block. The mechanical advantage is therefore six. If the lower block is raised one foot, the force at P will move through six feet, and a given force at P, say 100 lb., will therefore raise a weight at W six times as great, or 600 lb.

EXERCISES

1. What force is necessary to raise a weight of 450 lb. by an arrangement of six pulleys in a block and tackle?

Arrange a block and tackle to lift 500 lb. if only 50 lb. is available as the power.

2. Find the force which will support a weight of 720 lb. with three movable pulleys, arranged as in Fig. 164.

If 50 lb. is the available force, how many movable pulleys arranged as in Fig. 164 are necessary to lift 750 lb.? 1550 lb.?

3. A man weighing 145 lb. raises a weight of 448 lb. by a system of four movable pulleys arranged as in Fig. 164. What is his pressure on the ground as he raises the weight? *Ans.* 117 lb.

4. How many movable pulleys are required in a block and tackle if 42 lb. is available with which to raise 336 lb.?

207. Strength of Materials. In all farm buildings and farm implements the materials used are subjected to certain stresses and strains. The materials may be classified according to their physical structure under three headings: (1) wood; (2) metal; (3) stone, brick, cement, mortar, concrete.

The kind of material used, and its design, must meet the demands made upon it without breaking. Two prime considerations in the use of materials for machines and buildings are *the size and shape of the material*, as when used as a joist or a reaper tongue; and *the manner in which it is loaded.*

The ***strength of a beam*** is measured by the load it can safely withstand. The ***safe load*** which a beam will carry varies as each of the following factors.

(1) *Directly as the working fiber strength of its material.* Each kind of material thus is given *a rating or a constant.* This constant provides also for a reasonable ***factor of safety.*** In the table presented below, § 208, this factor of safety is six. This means merely that the safe load is one sixth of the breaking load.

(2) *Directly as the breadth times the square of the depth for beams with rectangular cross-section.* Wooden beams are usually of this form.

(3) *Inversely as the length.*

Thus we may write

$$\text{safe load} = \frac{\text{constant (breadth)(depth)}^2}{\text{length}}. \tag{7}$$

208. Safe Loads for Beams under Bending Stress. There are at least four common ways of loading beams. We present the formulas for safe loads in these four cases, with the following notation.

S = safe load in pounds, d = depth in inches,
b = breadth in inches, L = span in feet,
A = constant depending on the material.

The value of A is given in the following table for some common materials.

Material	Const. A	Material	Const. A
Yellow Pine . . .	100	White Oak . . .	75
White Pine . . .	60	Castiron	167
Hemlock	50	Steel	889
Chestnut	55	Limestone	8

Case I. *Beam supported at one end and loaded at the other* (Fig. 166).

$$S = \frac{bd^2A}{4L}. \tag{8}$$

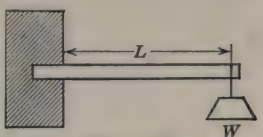


Fig. 166

Case II. *Beam supported at one end and uniformly loaded* (Fig. 167).

$$S = \frac{bd^2A}{2L}. \tag{9}$$

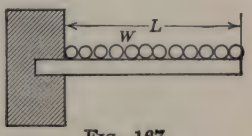


Fig. 167

CASE III. *Beam supported at both ends and loaded in the middle* (Fig. 168).

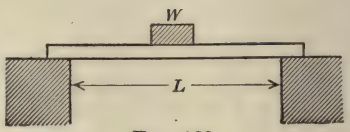


FIG. 168

$$(10) \qquad S = \frac{bd^2A}{L}.$$

CASE IV. *Beam supported at both ends and uniformly loaded* (Fig. 169).

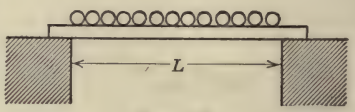


FIG. 169

$$(11) \qquad S = \frac{2\ bd^2A}{L}.$$

EXERCISES

1. A white oak timber 2 in. in width is to be used to support a hayfork and carrier at the end of a barn. How deep must the timber be to carry safely a load of 1200 lb. applied 3 feet from the single support?

Ans. $d = 9.8'' = 10''$ piece to be used.

2. Compare a $2'' \times 10''$ and a $3'' \times 8''$ timber as to cost and carrying capacity when used as a beam, edgewise. Do the same when used flatwise.

3. A steel beam $1\frac{1}{8}''$ wide is to be used to support a hayfork and carrier at the end of a barn. How deep must it be to carry safely a load of 1200 lb. applied 3 ft. from the single support?

4. What should be the maximum distance between supports for a $2'' \times 10''$ hemlock beam to safely sustain a middle load of 1000 lb.?

5. What length of beam could be used in Ex. 4, if the load of 1000 lb. is uniformly distributed?

6. Find the ratio of the safe loads of two beams of wood, one being 10′ long, $3'' \times 12''$ in cross-section and having its load in the middle, and the other 8′ long, $2'' \times 8''$ in cross-section with its load uniformly distributed.

7. What should be the breadth and depth of a yellow pine rectangular beam 20′ between the two end supports to carry safely a load of 1500 lb. at its center?

[HINT. The selection must be governed by commercial sizes, as the dimensions of wooden beams are not fractional.]

209. Work. ***Work****, in its scientific sense, is done only by a force in overcoming resistance.* The mere push or pull of the force is not work. In order that work may be done, the point of application of the force must *move.*

The work done is equal to the force applied multiplied by the distance through which it acts. (See § 201.) That is,

(12) $$\textbf{\textit{Work}} = \textbf{\textit{Force}} \times \textbf{\textit{Distance}}.$$

To compute the work done multiply the force in pounds by the distance in feet. The product is expressed in ***foot-pounds.*** ***A foot-pound of work*** is said to be done when a *force of one pound* moves its point of application through a *distance of one foot* in the direction of the force.

EXAMPLE. How much work is done when a force of 20 pounds moves its point of application 40 feet in the direction of the force?

A force of 20 pounds acting through 40 feet is the equivalent of 800 pounds acting through 1 foot, *i.e.* 800 foot-pounds of work is said to be done.

EXERCISES

1. How much work is done by a man weighing 150 pounds in climbing to the top of a barn 40 feet high? How much is done by a man weighing 120 pounds in walking up a hill 40 feet high? How much by a horse weighing 1200 pounds in walking up a hill 100 feet high?

2. Twenty pounds will pull a certain cart on a level road. How much work is done by a boy in pulling it a mile on this road?

3. The draft of a certain plow (*i.e.* the force required to pull it) is 500 lb. How much work is done in plowing a furrow 100 rods long?

How much work is done by this same plow in plowing a piece of land 60 rods long and 6 rods wide if the furrow slice is 12 inches wide?

4. What is the draft of a wagon, if 400,000 foot-pounds of work is done by a horse in drawing it a mile?

5. A baseball weighing $5\frac{3}{16}$ oz. was dropped from one of the windows at the top of Washington's monument and caught 4 ft. from the ground by a professional catcher. Compute the work done by the ball, assuming the window to be 545 feet from the ground.

6. A certain gas engine has a $4\frac{1}{8}$-inch bore, a $5\frac{1}{2}$-inch stroke, and makes 378 strokes per minute. How much work is done per minute if the average pressure against the piston head is 15 lb. per square inch?

210. Horse-power. If a man cuts 100 shocks of corn in a day of 10 hours, he does it at the *rate* of 10 shocks per hour. A horse which covers 21 miles in 3 hours has made the distance at the *rate* of 7 miles per hour. A belt which moves 50 feet per second has a *rate* of 3000 feet per minute. A horse which trots a mile in 2 minutes is covering the distance at the *rate* of 44 feet per second. If a force of 20 pounds acts through a distance of 300 feet for 5 minutes, it does 6000 foot-pounds of work but does it at the *rate* of 1200 foot-pounds per minute. These illustrations will make clear the following definition.

A ***horse-power*** *is the unit for the* ***rate*** *at which work is done.* This unit is 33,000 foot-pounds per minute, *i.e.* one horse-power means that a body is moved by a force of 33,000 pounds in the direction of the force one foot in one minute.

The value of this unit was adopted by JAMES WATT as the result of experiments with strong dray horses in England. He was aware, however, that it is in excess of what can be done by an average horse over a full working day.

211. Working Day for a Horse. Tests show that a steady pull equal to one tenth of the weight of the horse, for a ten-hour daily service, at the walking rate of 2.5 miles per hour, is an average of effective service.

One horse-power is developed by a horse pulling with a force of 150 pounds $2\frac{1}{2}$ miles per hour, since this means

$$\frac{5280 \times 2\frac{1}{2}}{60} = 220 \text{ feet per minute,}$$

and 150 pounds 220 feet per minute is 33,000 foot-pounds per minute, or one horse-power.

However, this power will not be developed by the average horse at the present time, since to draw 150 pounds $2\frac{1}{2}$ miles per hour and 10 hours per day, a horse should weigh from 1400 to 1800 pounds. This is larger than the ordinary horse used in America.

The average horse will pull about 120 pounds during the average working day.

Hence, the average horse will develop

$$\frac{5280 \times 2\frac{1}{2} \times 120}{60} = 26{,}400 \text{ foot-pounds per minute}$$
$$= \tfrac{4}{5} \text{ horse-power.}$$

EXERCISES

1. A horse draws a load having a draft of 150 pounds at the rate of 3 miles per hour. What horse-power is he developing? At this rate should this horse be required to work full time?

2. How much power is a horse developing when walking 4 miles per hour and pulling 75 pounds on his traces? 120 pounds?

3. A horse walks 2.5 miles per hour and pulls 120 pounds on his traces. How *fast* should he walk to develop the same power when the draft is increased to 160 pounds?

4. How much power does a 1600-pound horse develop when walking 2.5 miles per hour if he pulls one tenth of his weight? A 1200-pound horse? A 1000-pound horse?

5. If a horse weighs 1300 pounds and is pulling one tenth of its weight, how fast should it walk to develop one horse-power? If this horse walks 3 miles per hour, what horse-power is developed?

6. What should be the weight of a horse which develops 1.3 horse-power in walking 3 miles per hour and pulling about one tenth of its weight?

7. What power should be provided, allowing 25% loss of power due to friction, to unload hay at the rate of one half ton per minute if raised to a height of 30 feet?

8. How much power is necessary, if $\frac{4}{5}$ of it is available, to raise corn in an elevator to a height of 50 feet at the rate of 3000 bushels per hour? (1 bushel of corn = 70 pounds.)

9. In filling a water tank 60 feet high, the water is raised at the rate of 200 gallons per minute. What horse-power engine should be provided, granting that only $\frac{2}{3}$ of the power developed is available, and allowing in addition a 10% margin of safety for extraordinary pulls?

10. The maximum load on a hayfork is 800 pounds. What horse-power is required to raise this load 40 feet in 1 minute? How many horses each weighing 1000 pounds would be required to furnish this power if they walk 3 miles per hour and if only $\frac{1}{4}$ of the power applied is available?

11. An elevator has a capacity of 1500 pounds. What power is necessary to operate this elevator through 200 feet in 2 minutes?

12. Tests have shown that for the old-fashioned type of walking plow a furrow slice 8×13 inches requires a draft of 673.3 pounds. Find

(1) The horse-power required to draw this plow 2 miles per hour.

(2) The weight of each of three horses to pull this plow 2 miles per hour and 10 hours per day.

13. For a modern walking plow, a furrow slice 8×13 inches requires about 400 pounds draft. Find

(1) The rate at which the three horses of the previous example may walk in drawing this plow 10 hours per day.

(2) The horse-power required to draw this modern plow $2\frac{1}{2}$ miles an hour.

212. Draft. *Draft* is the amount of force required to move an object. The fundamental principle involved in overcoming resistance or in doing work is as follows.

The force doing the work when multiplied by the distance through which it moves is equal to the resistance multiplied by the distance through which it moves. (See § 201.) That is, in the case of raising a weight against the force of gravity,

$$\textit{Force} \times \textit{Force Distance} = \textit{Weight} \times \textit{Weight Distance}.$$

The amount of increase or decrease of draft up or down a grade does not depend upon the draft on the level but depends only upon the weight of the load and upon the slope of the grade. For grades less than 15% the horizontal

distance is approximately equal to the actual length along the grade, *i.e.* the hypotenuse is approximately equal to the base of the right triangle.

EXAMPLE. What force is required to pull a load of 60 pounds along a 10% grade, neglecting friction?

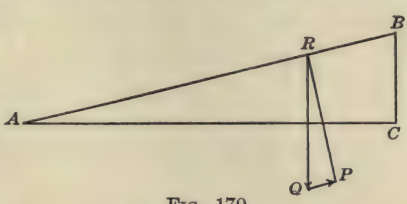


FIG. 170

A 10% grade means 10 feet rise in 100 feet. The force P moves horizontally 100 feet and the weight rises 10 feet since $RQ = 60$ pounds, $BC = 10$ feet, and $AB = 100$ feet $= AC$ approximately for grades of low per cent. Therefore,

$$P \times 100 = 60 \times 10, \quad \text{or} \quad P = 6 \text{ pounds.}$$

Also, from similar triangles

$$\frac{QP}{QR} = \frac{BC}{AB}, \quad \textit{i.e.} \quad \frac{QP}{60} = \frac{10}{100},$$

and therefore QP, parallel to the incline, is equal to 6 pounds. This is the force necessary to keep the object from sliding down the incline, neglecting friction.

But this amounts finally to merely multiplying the load by the grade per cent to obtain the increased or decreased draft up or down a grade. Hence we have the following rule.

***For draft up a grade**, add to the draft on the level a quantity obtained by multiplying the load by the grade per cent.*

***For draft down a grade**, subtract from the draft on the level the grade per cent of the total load.*

EXAMPLE. The draft of 4000 pounds on a certain level road is 250 pounds. What is the draft on a 3% grade?

3% of 4000 lb. = 120 lb.

Therefore 250 lb. + 120 lb. = 370 lb., the draft *up* a 3% grade,

and 250 lb. − 120 lb. = 130 lb., the draft *down* a 3% grade.

EXERCISES

1. A total load of 3700 pounds has a draft of 108 pounds per ton on a certain level road. What is the draft up a 3% grade? Down a 5% grade? Up a 13% grade? Down a 15% grade?

2. A total load of 4500 pounds has a draft of 560 pounds up a 3% grade. What is the draft per ton on the level? What is the draft down a 5% grade for this same load? Up a 12% grade?

3. A load of 2600 pounds has a draft of 155 lb. on a level road. What is the per cent grade down which the draft is zero? What would be the draft up this grade? How many degrees in the angle of elevation for this grade?

4. At what rate may a team of 1350-pound horses draw a total load of two tons on a level road, if the total draft is 165 lb.?

5. What should be the size of a two-horse team to draw a load of 2 tons up a 4% grade 3 mi. per hour, if the draft on the level is 225 lb.?

CHAPTER XV

COMPOSITION AND RESOLUTION OF FORCES

213. Graphic Representation of Forces. See § 5. A force to be completely described must have a *point of application*, a *direction*, and a *magnitude*.

Since a segment of a line may have all of these, it may be used to represent a force.

In Fig. 171, let P be a force applied at O and in the direction OP. Lay off on OP a number of units equal to the magnitude of the force. When the arrowhead is affixed to indicate the *sense* in the line of motion, the force is completely determined. For example, if a force of 10 lb. is represented on a drawing by 1″, then 30 lb. on the same drawing should be represented by 3″.

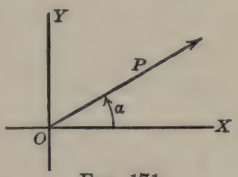


FIG. 171

214. Composition of Motions and of Forces. Parallelogram of Forces.

The ***resultant*** of two or more forces is that force which singly will produce the same effect as the forces themselves when acting together.

The ***components*** of the resultant are the individual forces which would together produce the same effect as the resultant.

The composition of motions, or of forces, or of velocities, is merely the process of finding their resultant. The following cases arise.

FIRST. *If two or more forces act in the same straight line, the resultant force is equal to their algebraic sum, and acts in the same line.* If the forces are not all in the same sense, either sense may be selected as positive and the other as negative.

SECOND. *If the forces do not act in the same straight line, the* ***parallelogram of forces*** *is used to find their resultant, as follows.*

When a body is acted upon by two forces, we draw two line-segments, AD and AB, Fig. 172, to represent the forces, as in § 213. If we complete the parallelogram $DABC$ and draw the diagonal AC, that diagonal represents the resultant of the two given forces both in direction and in magnitude.

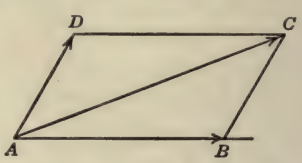


FIG. 172. — PARALLELOGRAM OF FORCES

Velocities may be combined to find a *resultant velocity* in precisely the same manner.

215. Computation of the Resultant of Two Given Forces. Let the two forces be F_1, F_2 and be represented by AB and AD, respectively (Fig. 173). Complete the parallelogram $DABC$, as in § 214, and let $R = AC$ be the resultant. Let θ denote the angle between the forces, and let α denote the angle between F_1 and the resultant. Project BC on AB. Then we have

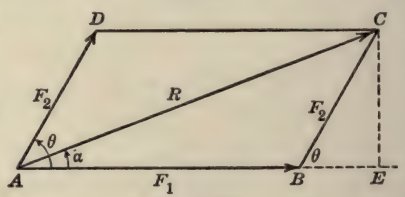


FIG. 173

(1) $BE = F_2 \cos \theta,$

(2) $EC = F_2 \sin \theta,$

(3) $AE = F_1 + F_2 \cos \theta,$

and

$$R = \sqrt{\overline{AE}^2 + \overline{EC}^2}, \tag{4}$$

$$\tan \alpha = \frac{EC}{AE}, \quad \text{and} \quad R = \frac{EC}{\sin \alpha}. \tag{5}$$

From these formulas, the resultant R and its direction α may be computed.

EXERCISES

1. Three forces of 5 lb., 3 lb., and 2 lb., respectively, act upon a point in the same straight line in one sense, and two other forces of 8 lb. and 9 lb. act in the same line in the opposite sense. With what force and in which direction will the point move?

Draw the conditions to the scale 1 lb. $= \frac{1}{4}''$.

2. Two forces whose magnitudes are as 3 to 4 acting on a point at right angles to each other produce a resultant of 30 lb. What are the component forces? What is the direction of the resultant?

Ans. 18 lb. and 24 lb.

3. Let ABC be a triangle, and D the middle point of the side BC. If the three forces, represented in magnitude and direction by AB, AC, and AD, act upon the point A, find the direction and magnitude of the resultant. *Ans.* $3\,AD$ in the direction AD.

4. What single force would be sufficient to counteract a vertical force of 5 pounds and a horizontal force of 12 pounds acting on a body at the same instant? What would be the *direction* of this resultant?

Ans. 13 pounds, 22° 37′ with the horizontal.

5. Find the magnitude and the direction of the resultant of two forces of 30 pounds and 40 pounds acting on a body at an angle of 60° with each other.

[Hint. In the accompanying figure, find BE and EC. Then CA may be found from the right triangle AEC.]

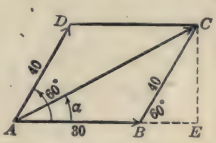


Fig. 174

6. A river is flowing at the rate of 3 mi. per hour. How should a man, rowing a boat 5 miles per hour, point the boat up the river in order that his course may be directly across the river? What would be his *rate* directly across? What is the *time* across if the river is 2 mi. wide?

7. A man rows a boat across a river at the rate of 3.5 miles per hour. The river flows at the rate of 4.8 miles per hour. Find the speed of the boat. Where will it land along the opposite bank if the river is 2 miles wide?

8. Given $F_1=37.6$, $F_2=58.7$, $\theta=58°\ 20'$ (Fig. 173, § 215), find R and α.

9. Given $F_1=37.6$, $F_2=58.7$, $\theta=71°\ 40'$, find R and α.

10. Given $F_1=62.7$, $F_2=46.8$, $\theta=90°$, find R and α.

11. From an auto going 25 mi. per hour a man throws a stone at the rate of 60 ft. per second and at an angle of 45° with the line of motion of the auto. Find the speed and direction of the stone.

12. Two forces of 4 lb. and $3\sqrt{2}$ lb. act at an angle of 45°, and a third force of $\sqrt{42}$ lb. acts at right angles to their plane at the same point. Find their resultant. *Ans.* 10 lb.

216. Resolution of Forces. By the *resolution of forces* is meant the process of finding the components of given forces in two given directions. We saw in § 215 that two forces, F_1 and F_2, acting at a point are equivalent to a single force, R, the resultant. It is evident then that the single force, R, acting along the diagonal AC of the parallelogram, Fig. 173, can be replaced by the two forces, F_1 and F_2, represented in magnitude and direction by two adjacent sides of the parallelogram.

The given total force AC may be used as the diagonal for an infinite number of parallelograms. For we can draw through A two lines AD and AB in any given directions. It follows that a single force may be resolved into components in any two given directions.

217. Rectangular Components of a Force. The most convenient components into which a force can be resolved are those whose directions are at right angles to each other. Thus let OX and OY be any two lines at right angles to

each other and P, Fig. 175, any force acting at O in the plane XOY. Completing the rectangle $OMPN$, we find the components along the two axes:

(6) x-component $= OM = P \cos \alpha$,

(7) y-component $= ON = P \sin \alpha$,

where α is the angle which the direction of P makes with the x-axis.

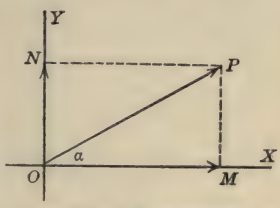


FIG. 175. — RECTANGULAR COMPONENTS OF A FORCE

The x- and the y-components of a force are called its ***rectangular components.*** They are usually the ***horizontal*** and the ***vertical components.*** See also § 5.

Unless otherwise expressed, the components of a force will mean its rectangular components on a horizontal line and a vertical line.

Velocities may be resolved into component velocities in precisely the same manner.

EXERCISES

1. The tension in each of four traces by which two horses are drawing a plow is 50 lb. What are the horizontal and vertical components of the total force moving the plow if the traces make an angle of 14° with the horizontal?

2. A ball rolls along the diagonal of a $6' \times 18'$ wagon bed at the rate of 10 feet per second. What is the speed of the ball when the wagon is moving forward at the rate of 12 feet per second?

3. Sleet is falling with a vertical speed of 16 feet per second. What is the velocity of the wind when the sleet falls at an angle of 60° with the horizontal? What is the wind's velocity when the sleet falls at an angle of 45° with the horizontal?

Ans. 6.3 mi. per hour; 11 mi. per hour, approximately.

4. What must be the tension in a rope to support on a smooth ring a pig weighing 100 lb. between two posts 12 feet apart if the rope is allowed to sag 3 feet at the center? What is the tension when the sag is 2 feet? *Ans.* 112 lb., 158 lb.

5. A weight W lies on a smooth inclined plane which makes an angle α with the horizontal. Show that the components of W along and perpendicular to the plane are

$$P = W \sin \alpha, \quad R = W \cos \alpha,$$

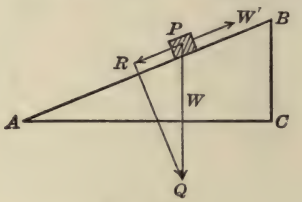


FIG. 176

where R is the reaction against the plane and P is the force acting parallel to the plane to keep the weight from slipping.

6. A weight of 10 lb. is held in position on a smooth inclined plane (inclination α) by a force of 2 lb. acting up the plane and by a force acting horizontally on the weight. Determine this horizontal force if $\sin \alpha = \frac{3}{5}$.

[HINT. Consider the components of all of the forces in the direction of the plane.] *Ans.* 5 lb.

7. A farmer sets three posts in the ground so as to form an equilateral triangle. A rope is passed around them the tension in which is 6 lb. Find the pressure on the posts. *Ans.* $6\sqrt{3}$ lb.

8. A barn is equipped with a hayfork, pulleys, etc. The rope leading from the roof pulley to the floor pulley makes an angle of 70° with the horizontal. In drawing up hay the tension in the rope is 800 lb. and the rope beyond the floor pulley makes an angle of 10° with the horizontal. Find the vertical force necessary to hold the floor pulley to its fastening. Use scale diagrams as an approximate check. Consider the entire rope at all times in the same vertical plane. Study the following cases:

FIG. 177

(*a*) When the draft end of the rope is to the *left* of the floor pulley and 10° *below* the horizontal.

(*b*) When it is to the *left* of the floor pulley but 10° *above* the horizontal.

(*c*) When it is to the *right* of the floor pulley and 10° *above* the horizontal.

9. A wind is blowing nearly from the west, as shown in the accompanying figure, against the sail, AB, of a vessel going due north. What part of this wind force, WC, tends to drive the vessel north?

We may regard the wind force, WC, as the resultant of two forces, DC, perpendicular to the sail and tending to drive the vessel nearly northeast and WD parallel to it. But DC may be resolved into DE and EC, one of which tends to push the vessel sidewise, the other propels it northward. The side thrust is largely counteracted by the water.

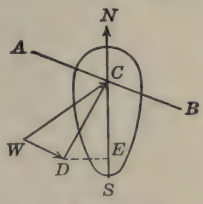


FIG. 178

10. Show how the rigging of the vessel in the previous exercise may be shifted so that the same wind will propel the vessel south.

Will the component propelling the vessel south be equal to the one propelling it north?

11. Two equal rafters support a weight W at their upper end. Find the compression in each. Let the length of each rafter be a and the horizontal distance between their lower ends be $2b$, Fig. 179.

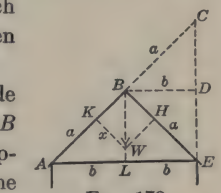


FIG. 179

Let BW represent in direction and magnitude the weight W. From W draw WH parallel to AB and WK parallel to EB. Then BK and BH represent the components of the force BW along the rafters BA and BE (§ 216). Draw CE parallel to BL and extend AB to C. From the similar triangles BWK and CEB,

$$\frac{BK}{W} = \frac{a}{2\sqrt{a^2-b^2}}, \quad i.e. \quad BK = \frac{aW}{2\sqrt{a^2-b^2}}.$$

Find also the tension in the tie-rod along b, by taking the component of BK on the line AE.

12. Find the thrust in the equal rafters and the tension in the tie-rod for rafters 10 ft. long in a barn 16 ft. wide due to a total weight of 100 lb.

What are the thrust and the tension for rafters 9 ft. long in a barn 16 ft. wide?

What is the effect of short rafters?

218. Triangle of Forces. The resultant of two forces F_1 and F_2, Fig. 180, may be obtained more easily by drawing AB to represent the magnitude and direction of one force and BC, similarly, for the other force. Then AC represents the resultant. The force CA (in direction from C to A) is opposite in direction but equal in magnitude to the resultant of F_1 and F_2.

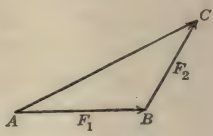


FIG. 180

If three forces be represented in magnitude and direction by the sides of a triangle, taken in order, they will be in equilibrium.

219. The Simple Jib Crane. For moving heavy weights, a machine called a ***crane*** is much used. It consists of a ***crane post*** AB, a ***tie-rod*** BC, a ***jib*** CA. When loaded with a weight W attached at C, the point C is in equilibrium under the action of the three forces

(*a*) *the weight* W,
(*b*) *the pull in the tie-rod,*
(*c*) *the thrust in the jib.*

These form a triangle of forces in which any one of the forces is the resultant of the other two, and they may be represented on a convenient scale by the three sides of a triangle, taken in order. In Fig. 181, CW represents the weight to scale. Draw WD parallel to AC and extend BC to D. Then on the same scale CD represents the pull in the tie-rod and WD the thrust in the jib. The sense of the forces is obtained by following around the triangle in order, from C to W to D to C.

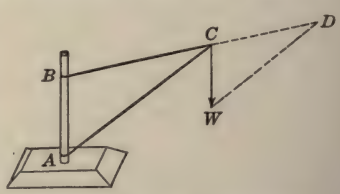


FIG. 181. — THE JIB CRANE

EXAMPLE. A weight of 10 tons is raised by a crane whose jib is 30′ long and stands at an angle of 45° with the crane post. Find the thrust in the jib if the tie-rod is attached 20′ from the bottom of the crane post. Since WCD is similar to ABC,

$$\frac{WD}{10}=\frac{30}{20}.$$

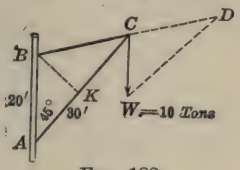


FIG. 182

Thus $WD=15$ tons, the thrust in the jib. To compute CD we must know BC. This is found from the triangle ABC as follows: Draw BK perpendicular to AC.

Then $BK=20\sin 45°=14.1$, $AK=20\cos 45°=14.1$,

$KC=30-14.1=15.9$.

Therefore

$$BC=\sqrt{(14.1)^2+(15.9)^2}=21.3$$

Since

$$\frac{CD}{10}=\frac{BC}{20}=\frac{21.3}{20},$$

it follows that

$CD=10.6$ tons, the pull in the tie-rod.

When the weight is supported by a rope passed over pulleys at C and B and on to a drum on the crane post, *the tension in the tie-rod is made less* by an amount equal to the tension in the rope which supports the weight. This tension is the weight itself.

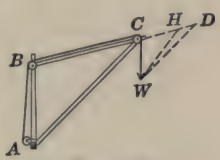


FIG. 183

Graphically, if CW represents the weight and therefore the tension in its rope, CD is the whole tension in both tie-rod and rope. Mark off CH on CD and equal to CW. Then HD must be the tension in the tie-rod in this case.

EXERCISES

1. A weight of 12 tons is suspended rigidly at the end of a jib 20 feet long by a tie-rod 15 feet long. Find the thrust in the jib and the tension in the tie-rod if the tie-rod is attached to the crane post.

(*a*) 10 feet from its lower end. *Ans.* 24 tons; 18 tons.

(*b*) 13 feet from its lower end.

(*c*) 15 feet from its lower end.

(*d*) 20 feet from its lower end.

2. In Ex. 1, find, for each of the different cases, the tension in the tie-rod when the weight is supported by a rope which passes over a pulley at C and is led *parallel to the tie-rod* over a pulley at B and on to a drum on the crane post.

What happens in cases (c) and (d)?

3. If $AB = 12'$, $AC = 8'$, and $W = 5$ tons, supported rigidly at C, Fig. 181, find the thrust in the jib and the tension in the tie-rod when $BC =$ (a) 8 feet; (b) 10 feet; (c) 12 feet; (d) 14 feet.

4. Discuss the previous exercise when the weight is supported by a rope led over a pulley at C, parallel to the tie-rod, and over a pulley at B.

5. The load suspended rigidly from the end of a jib is 5 tons. The compression in the jib is 12 tons and the inclination of the jib to the horizontal is 60°. Find the tension in the tie-rod.

6. Discuss Ex. 5 when the jib is inclined to the horizontal at an angle of (a) 45°; (b) 30°.

7. A rod AB, 30″ long, hinged at A and carrying a weight of 150 lb. at B, is held horizontally by a rope BC making an angle of 45° with AB. Find the tension in the rope.

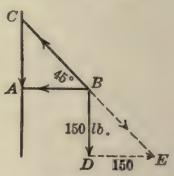


FIG. 184

Let the segment $BD = 150$ lb. in the triangle BDE similar to ABC, in the accompanying figure. Then BE on the same scale is the tension in BC. What is DE? *Ans.* Tension $= 150\sqrt{2}$ lb.

8. Solve Ex. 7 if BC makes an angle of 135° with AB. Also if it makes an angle of 120° with it.

[HINT. Let CB represent 150 lb. To find $x = BO$.]

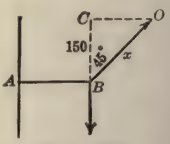


FIG. 185

9. A weight of 24 lb. is suspended by two flexible strings one of which is horizontal and the other is inclined at an angle of 30° with the vertical. What is the tension in each string? *Ans.* $8\sqrt{3}$ lb.; $16\sqrt{3}$ lb.

10. Hay is pulled up at the end of a barn into a mow by fork, pulleys, etc. The top pulley is attached to a beam AB fastened at A to the end of the barn. The beam is held horizontal by a rope attached at B and making an angle of 45° with AB. What is the tension in the rope if the top pulley is fastened to the beam 4′ from A and if AB is 6′ long, considering a load of 300 lb. on the pulley?

Find also the thrust in the beam against the end of the barn.

[HINT. Determine first what weight at B will produce the same moment as 300 lb. at the pulley.]

11. A rod 10″ long, hinged at one end, supports a weight of 4 lb. 3″ from the attached end. The other end is free to move but is held horizontal by a string attached to this end and making an angle of 120° with the rod. Find the tension in the string. Also the pull in the rod.

[HINT. Find first the equivalent weight at the end of the rod.]

Solve also when the weight of 4 lb. is attached to the rod 6″ from the hinged end.

12. A man weighing 200 lb. stands at the apex of a roof whose equal rafters are 10 ft. long on a building 16 ft. wide. Find the weight tending to spread the walls and the thrust in the rafters.

What would be the effect of shortening the rafters for the same width of the building?

13. When a roof, as the one in Ex. 12, is covered uniformly with snow, what part of the weight of the snow tends to spread the walls?

220. Resultant of Several Forces in a Plane Acting at a Point.

FIRST METHOD. The resultant may be found by repeated application of the triangle of forces. Let F_1, F_2, F_3, F_4 be forces acting upon a particle at O. The resultant of F_1 and F_2 is OB, Fig. 186; of OB and F_3 is OC; of OC and F_4 is OD. This process may be continued for any number of forces acting in the plane at a point. Thus OD is in magnitude and direction the resultant of the given forces.

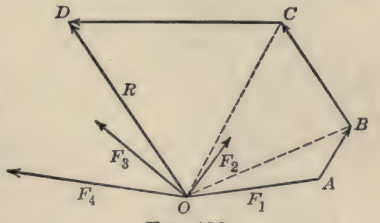


FIG. 186

But this amounts to omitting the dotted lines and drawing AB, in sense and direction, equal to F_2, BC similarly

equal to F_3, and CD similarly equal to F_4, and connecting D to O. This gives rise to a closed polygon called the ***polygon of forces.***

The polygon of forces perhaps reminds one of the diagram to scale of a closed survey in which the last course which closes the survey would be the resultant. This idea gives rise to a second method of finding the resultant.

Second Method. Let forces F_1, F_2, F_3, F_4, etc., in the same plane act on a particle at O. Refer the forces to a pair of coördinate axes OX and OY. In surveying, these axes are the north-south line and the east-west line. Resolve each force into its rectangular components with reference to the axes as shown in Fig. 187. For example,

$$OA = F_1 \cos \alpha_1,$$
$$OB = F_1 \sin \alpha_1,$$

etc., for each of the other forces.

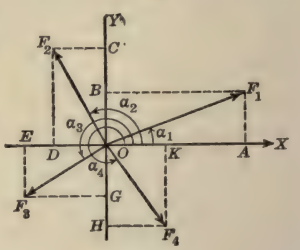


Fig. 187

If a component acts upward or to the right it is positive; if downward or to the left it is negative. For example, OD and OG are negative.

Let us denote by X the algebraic sum of the x-components, and by Y the sum of the y-components. The system of forces is now reduced to a system of two forces at right angles to each other, whose resultant has a magnitude

$$R = \sqrt{X^2 + Y^2}. \tag{8}$$

The direction of the resultant, α, is given by

$$\tan \alpha = \frac{Y}{X} \tag{9}$$

where α is the angle between the positive direction of R and the positive direction of the x-axis. If X and Y are

plotted properly, the quadrant in which α lies will be definitely determined.

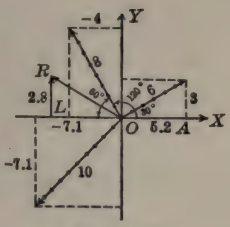


FIG. 188

EXAMPLE. Forces of 6, 8, 10 lb. act on a particle at angles of 30°, 120°, 225°, respectively, with the horizontal. Find the magnitude and direction of the resultant.

The details for taking the rectangular components may be arranged as follows.

FORCES	x-comp.		y-comp.	
	+	−	+	−
6	5.2		3	
8		4	6.9	
10		7.1		7.1
Sums	5.2	11.1	9.9	7.1

Hence $X = -5.9, \quad Y = +2.8$

Lay off on the diagram $X = OL = -5.9$ and $Y = LR = 2.8$. Then

$$OR = \sqrt{2.8^2 + 5.9^2} = 6.5,$$

and $$\tan \alpha = \frac{2.8}{-5.9} = -.4746$$

Look up in the table $\tan \alpha' = .4746$ and subtract from 180°.

Thus, $\alpha' = 25° 23'$ with the negative end of x-axis or

$\alpha = 154° 37'$, the direction of R with the positive x-axis.

EXERCISES

1. Find graphically the resultant of the forces in the example of § 220 by using the polygon of forces.

2. Forces of 2 lb., 3 lb., 4 lb. act along the sides of an equilateral triangle taken in order. Find the magnitude and the direction and the line of action of the resultant.

Find the resultants of the following systems of forces. Check graphically.

3. 100 lb. acting at an angle of 0° with the horizontal, *i.e.* (100 lb., 0°), and similarly (50 lb., 60°), (200 lb., 180°). *Ans.* (86.6 lb., 150°).

4. (4 lb., 60°), (5 lb., 135°), (6 lb., 210°), (8 lb., 330°).

5. (9 lb., 0°), (7 lb., 50°), (5 lb., 150°), (3 lb., 225°).

6. (250 lb., 0°), (270 lb., 33°), (160 lb., 114°).

Ans. (502 lb., 55°).

7. (270 lb., 0°), (132 lb., 54°), (215 lb., 82°), (138 lb., 330°).

Ans. (558 lb., 27°).

8. (35 lb., 0°), (24 lb., 132°), (54 lb., 239°).

Ans. (297 lb., 253°).

9. (60 lb., 0°), (58 lb., 90°), (86 lb., 292°), (110 lb., 189°).

Ans. (42.3 lb., 247°).

221. Resultant of Two Parallel Forces. In Fig. 189, let AP, BQ represent two parallel forces P and Q acting at A and B respectively upon a body in the same direction. The resultant of P and Q is desired.

At A and B introduce $AN = BN'$, any two equal but opposite forces. Since their resultant is zero, the resultant of P and Q is in no way disturbed.

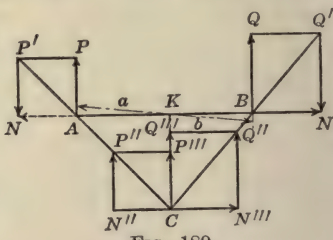


FIG. 189

At A draw AP' the resultant of AP and AN and similarly at B. Produce $P'A$ and $Q'B$ to meet at C.

At C construct the parallelograms of forces exactly equal to those at A and B respectively. Now $CN'' = CN'''$ and thus neutralize each other while the components CP''', and CQ''', equal respectively to P and Q, act in the line CK parallel to the lines of action at A and B. Thus, ***the resultant of P and Q is P+Q.***

Moreover, from similar triangles,

$$\frac{AP}{AN} = \frac{CK}{AK}, \quad \text{and} \quad \frac{BQ}{BN'} = \frac{CK}{BK},$$

Dividing these two equations, member for member, and remembering that $AN = BN'$, we have

$$\frac{AP}{BQ} = \frac{BK}{AK} = \frac{b}{a}.$$

Therefore, since AP represents the force P, and BQ represents the force Q, it follows that

$$\frac{P}{Q} = \frac{b}{a}, \quad i.e. \quad Pa = Qb. \tag{10}$$

A similar proof may be given for the case of unequal parallel forces acting in *opposite* directions. See also §§ 198, 214. Both results may be combined as follows.

The resultant of two parallel forces, acting in the same or opposite directions at the extremities of a rigid straight line, is parallel to the given forces, equal to their algebraic sum, and divides the given line into segments which are inversely proportional to the given forces.

EXERCISES

1. A lever 10 ft. long is used to raise a weight of 300 lb. Where should the fulcrum be placed if a man's weight of 150 lb. is available at the other end of the lever?

2. The draft of a plow is 365 lb. It is drawn by two horses hitched to a doubletree of which the end clevis holes are 15 in. and 19 in., respectively, from the middle hole. How much does each horse pull?

3. Solve Ex. 2 for a draft of p lb. and end clevis holes a in. and b in. from the middle hole.

4. A horse weighs 1356 lb. of which 795 lb. are on the front feet and 561 lb. on the hind feet. If the distance between the front and hind feet, as measured at their points of contact with the ground, is 45 in., where does the line of gravitation fall between the feet, *i.e.* where is the line of action of the resultant of these two parallel forces?

5. A horse weighing 1250 lb. sustains 700 lb. on his front feet. The distance between fore and hind feet is 48 in. Where is the line of gravitation?

6. If the line of gravitation for the horse in the previous example falls 25 in. to the rear of the front feet, what is the distance between front and rear feet?

7. Let six parallel forces proportional to the numbers 1, 2, 3, 4, 5, 6 act at the points $(-2, 0)$, $(-1, 0)$, $(0, 0)$, $(1, 0)$, $(4, 0)$, $(-3, 0)$, respectively. Find the resultant and its point of application.

8. On a straight rod AP there are suspended 5 weights of 5, 15, 7, 6, 9 lb. at the points A, B, C, D, P, respectively, for which $AB=3$ ft., $BC=6$ ft., $CD=5$ ft., $DP=4$ ft. Find the resultant and its point of application.

9. A scantling $4'' \times 4'' \times 16'$ is supported 1 ft. from its end. What weight will it lift at the short end due to its own weight when horizontal if it weighs 4 lb. per linear foot? Solve also when the scantling stands at an angle of 30° with the horizontal.

[HINT. The weight of the scantling acts at its center.]

10. A bar of uniform thickness, 10 ft. long and weighing 150 lb., is supported at its extremities in a horizontal position, and carries a weight of 400 lb. at a point distant 3 ft. from one extremity. Find the pressures on the points of support, *i.e.* at A and at D.

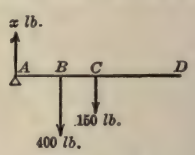


FIG. 190

[HINT. Let x = pressure at A, *i.e.* x lb. would just raise the lever AD. Taking moments about D, we have

$$x(10) \text{ ft.} = (400)7 \text{ ft.} + (150)5 \text{ ft.},$$

i.e. $x = 355$ lb. pressure at A.]

11. A bridge is 30 ft. long and weighs 15 tons. Find the pressure on the abutments when a wagon weighing 2 tons is one third of the way across. *Ans.* $8\frac{1}{6}$ tons; $8\frac{5}{6}$ tons.

12. The wheels of a wagon carry a load of 800 lb. What extra horizontal pull is necessary to draw the load over an obstruction 2″ high when the wheels are

(*a*) 48″ in diameter,
(*b*) 42″ in diameter,
(*c*) 36″ in diameter,
(*d*) 30″ in diameter?

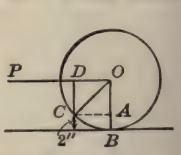


FIG. 191

[HINT. The weight acts in the direction OB. Take moments about C, *i.e.* $P \times DC = \text{Weight} \times AC$.] *Ans.* (*a*) 349 lb.

13. A carriage wheel whose weight is W and radius r rests upon a level road. Show that the pull P necessary to draw the wheel over an obstacle, of height h, is

$$P = W\frac{\sqrt{2rh-h^2}}{r-h}.$$

14. What height of obstacle may be cleared by the wheels of a wagon carrying a load of 800 lb. if the wheels are 36″ high and an extra pull of 150 lb. is available? *Ans.* .31″

15. Two horses weighing, respectively, 1100 lb. and 1500 lb. are hitched to a doubletree 38″ long. Where should the clevis hole be located so that each horse shall pull in proportion to his weight?

16. Figure 192 represents a doubletree with end holes A and B placed 36 inches apart and the center hole, C, 3 inches in front of the mid-point of AB. If the total draft of the load is 400 lb., what is each horse pulling when the doubletree is at an angle of 60° with the line of draft?

[HINT. Compute AH, HK, and KE. Then L_1 times AH must equal L_2 times HE. Why?]

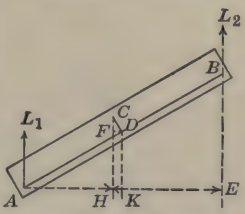


FIG. 192

17. Solve Ex. 16 when the angle made by the doubletree with the line of draft is 64° 33′.

18. Solve Ex. 16, if the middle hole C is 4″ in front of the center of AB.

19. Where should C be located so that the forces L_1 and L_2 are equal for any angle the evener makes with the line of draft? Where should it be so that the horse ahead pulls the greater load?

APPENDIX

TABLE I. Drainage. A fall of 6″ to 1′ for each 100′ is considered a good grade for farm drainage.

Let A = the number of acres to be drained.
d = depth in inches of water to be drained off in 24 hr.
t = diameter of tile in inches.
D = number of cubic feet of water discharged per second for a 1% grade.
x = the fall in inches per 100′.

The following table gives D for tiles of different sizes.

Tile diameter, t . .	4	6	8	10	12	15	20
Discharge, D . .	0.16	0.49	1.11	2.05	3.40	6.29	13.85

Then

$$A = \frac{2}{d} D\sqrt{12\,x}.$$

TABLE II. Weights of Produce.

(The figures represent minimum weights per bushel.)

Barley . . . . . 48 lb. (in Ore., 46 lb.; Ala., Ga., Ky., Pa., 47 lb.; Cal., 50 lb.; La., 32 lb.)
Beans (white) . . . . 60 lb.
Bran . . . . . 20 lb.
Buckwheat . . . . . 48 lb. (in Cal., 40 lb.; Ky., 56 lb.; Ida., N. D., Okl., Ore., S. D., Texas, Wash., 42 lb.; Kan., Minn., N. C., N. J., O., Tenn., 50 lb.)
Clover Seed . . . . . 60 lb. (in N. J., 64 lb.)
Corn (ear) . . . . . 70 lb., except in Miss., 72 lb.; in Ohio, 68 lb.; in Ind. after Dec. 1 and in Ky. after May 1, following the time of husking, it is 68 lb.

Corn (shelled)	56 lb. (in Cal., 52 lb.)
Corn (Meal)	50 lb. (in Ala., Ark., Ga., Ill., Miss., N. C., Tenn., 48 lb.)
Flax Seed	56 lb.
Grass Seed (blue)	44 lb.
Grass Seed (Hungarian)	50 lb.
Hemp Seed	44 lb.
Millet Seed	50 lb.
Oats	32 lb. (in Ida., Ore., 36 lb.; in Md., 26 lb;. in N. J., Va., 30 lb.)
Onions	57 lb.
Peas	60 lb.
Potatoes (white)	60 lb. (in Md., Pa., Va., 56 lb.)
Potatoes (sweet)	55 lb.
Rye	56 lb. (in Cal., 54 lb.; in La., 32 lb.)
Timothy Seed	45 lb. (in Ark., 60 lb.; in N. D., 42 lb.)
Turnips	55 lb.
Wheat	60 lb.

TABLE III. Pounds of Nitrogen; Phosphoric Acid; and Potash required per acre, as a minimum, to raise the following crops. Double the figures for maximum.

Cereal Crops. Barley: 12; 20; 25. Buckwheat: 15; 30; 35. Corn and sorghum: 10; 35; 30. Oats and rye: 12; 20; 30. Wheat: 12; 20; 12.

Garden Crops. Asparagus: 20; 30; 35. Cabbage and cauliflower: 40; 70; 90. Celery: 40; 50; 65. Cucumbers, muskmelon, pumpkin, squash, watermelon: 30; 50; 65. Egg plant: 40; 50; 90. Lettuce: 40; 50; 75. Onions: 45; 55; 80. Radishes: 15; 35; 45. Spinach: 15; 55; 40. Tomatoes: 25; 35; 40.

Grasses. Lawns: 20; 25; 30. Meadow, millet: 15; 30; 35. Pasture: 15; 30; 40.

Legumes. Alfalfa, clover: 5; 30; 40. Beans, peas: 5; 30; 35.

Orchard Crops, Small Fruits. Apples, pears, quinces: 8; 30; 50. Blackberries: 15; 30; 40. Cherries, plums: 10; 35; 45. Currants, gooseberries: 10; 25; 40. Grapes: 8; 30; 45. Nursery stock: 10; 25; 30. Peaches: 15; 40; 55. Strawberries: 25; 55; 70.

Roots, Tubers. Beets, turnips: 20; 25; 35. Carrots: 15; 35; 40. Horse radish: 15; 25; 35. Parsnips: 20; 55; 50. Potatoes: 30; 40; 65.

TABLE IV. Percentage of Nitrogen: Phosphoric Acid: Potash in Common Fertilizing Materials.

(Analyses vary considerably. The following figures represent rather high grade material.)

Nitrogenous Materials. Dried blood, 13 : 0 : 0. Sodium nitrate, 15 : 0 : 0. Ammonium sulphate, 20 : 0 : 0. In analysis and guarantees, ammonia = about .82 nitrogen.

Phosphatic Materials. Acid phosphate (superphosphate, dissolved stone, acid rock), 0 : 16 : 0. Basic slag (Thomas slag, iron phosphate), 0 : 15 : 0. Floats (calcium phosphate, phosphate of lime, raw phosphate, raw rock), 0 : 12 : 0. Acid bone black, 0 : 15 : 0.

Combined Nitrogenous and Phosphatic Materials. Acid bone, 2 : 20 : 0. Bone meal, 3 : 24 : 0. Fish scrap (fish guano, fish tankage), 8 : 7 : 0. Ground tankage (meat guano, slaughterhouse refuse), 7 : 9 : 0. Steamed bone, 2 : 26 : 0. Manure cake, 4 : 4 : 0.

Potash Materials. Muriate of potash (potassium chloride), 0 : 0 : 50. Kainit, 0 : 0 : 12. Sulphate of potash, 0 : 0 : 45.

Combined Phosphatic and Potash Materials. Wood ashes, 0 : 1 : 5. Dry ferns, 0 : .37 : 1.8.

Combined Nitrogenous, Phosphatic, and Potash Materials. Corn stalks, .5 : .33 : 1.67. Guano (Peruvian), 5 : 18 : 3. Cottonseed meal (cottonseed cake), 7 : 2.5 : 1.5. Farmyard manure, .5 : .5 : .5. Hen manure, dry, 2 : 2 : 1. Leaves, .7 : .15 : .3. Oat straw, .7 : .2 : 1.1. Pea straw, 1 : .3 : 1. Sheep manure, 2 : 1.5 : 1.5.

TABLE V. Units and Equivalents in Weights and Measures.

1 acre	43,560 sq. ft. = 160 sq. rd. = 10 sq. ch. = 40.47 ares (metric)
1 are (metric)	100 sq. meters = .01 hectare (ha.) = 119.6 sq. yd. = .02471 (1/40) acre
1 bag cement	.94 cu. ft. = 94 lb.
1 barrel	31½ gal. = 4⅕ cu. ft.
1 barrel cement	4 bags = 376 lb. = 3.76 cu. ft.
1 barrel flour	196 lb.
1 barrel refined oil	42 gal.
1 barrel salt	200 lb.
1 brick	2″ × 4″ × 8″
1 bushel (stroked)	2150.4 cu. in. = 1¼ cu. ft. = approximately 35.24 liters

1 bushel (heaped)	2688 cu. in. to 2747 cu. in.
1 bushel of coal	80 lb.
1 centimeter (cm.)	.3937 in. = 10 millimeters
1 chain	66 ft. = 100 links (li.)
1 cubit	18 inches
1 cubic centimeter water	1 gram (at 4° C.)
1 cu. in.	16.387 c.c.
1 cu. ft.	= 1/27 cu. yd. = .037 cu. yd. = .0283 cubic meter = 4/5 bu. = 7½ gal.
1 cu. ft. brickwork	18 to 22 bricks
1 cu. ft cement	100 lb.
1 cu. ft. water	62.43 lb. = 1000 oz. approximately
degrees Fahrenheit (F°)	$F° = \frac{9}{5} C° + 32°$
degrees Centigrade (C°)	$C° = (F° - 32°)\frac{5}{9}$
1 dram (avoirdupois)	1/16 oz.
1 fathom	6 feet
1 foot	12 inches = 30.48 cm.
1 foot (board measure)	a board 1 ft. square and 1 in. (or less) thick
1 furlong	40 rods
1 gallon	4 quarts = 3.7854 liters = 231 cu. in.
1 gallon 20% cream	8.4 lb.
1 gallon 22% cream	8.339 lb. = 1 gallon water
1 gallon milk (3¼%)	8.6 lb.
1 gallon skim milk	8.65 lb.
1 gallon water	8.339 lb.
1 grain (avoirdupois)	.065 gram
1 gram	15.432 grains = .03527 oz.
1 gross	12 dozen
1 hand	4 inches
1 hectare (ha.)	100 ares = 2.471 acres
1 hogshead	2 bbl. = 63 gal.
1 inch	2.54 cm. = 25.4 mm.
1 kilogram, or kilo (kg.)	1000 g. = 2.20462 lb.
1 kilometer (km.)	1000 meters = .62 mi.
1 knot	1 nautical mile per hour = 1.15 mi. per hour
1 labor (lä-bōr′)	1,000,000 sq. varas = 173 acres

1 league	3 nautical mi. = 3.46 mi.
1 league (Spanish unit of area)	25 labors = 6.8 sq. mi.
1 link	7.92 in. = .66 ft.
1 liter	1000 cc. = 244 cu. in. = 1.057 qt. (liquid) = .908 qt. (dry)
1 meter	39.37 in. = 3.3 ft.
.91 meter	1 yard
1 mil	1/6400 of a circumference = .001 radian, approximately
1 mile	5280 ft. = 8 furlongs = 320 rd. = 1.6 kilometers
1 nautical mile	1.153 mile = 1 minute on equator
1 millimeter	.039 in. = .1 cm.
1 oz. (avoirdupois)	28.35 grams
1 oz. (Troy)	31.1 grams
1 pace	3 feet
1 penny (English)	$.0203 (before 1914)
1 perch = 1 pole	1 rod = 16½ feet
1 perch of stonework	240 cu. ft.
π (pī)	3.1416 (See p. 28.)
1 pole	(see "perch")
1 pound (avoir.)	7000 grains = 16 oz. = .4536 kilogram
1 pound English money	20 shillings = $4.866 (before 1914)
1 pound (Troy)	5760 grains = 12 oz.
1 quart (liquid)	.95 liter (.9464)
1 quart (dry)	1.1 liter
1 radian	$180° \div \pi = 57.3° = 1000$ mils
1 rod	= 1 perch = 1 pole = 16½ feet
1 shilling (English)	$.243 (before 1914)
1 square foot	144 square inches
1 square inch	6.5 square centimeters
1 ton	2000 lb.
1 ton of hay	500 cu. ft. (varies from 450 cu. ft. to 550 cu. ft. depending on quality)
1 ton of coal	25 bu. heaped (2688 cu. in.) = 38⅘ cu. ft.
1 vara (vä'rä)	33⅓ in. in Texas 33 in. in Cal. and Mexico
1 yard	3 ft. = .91 meter

TABLE VI. American Experience Mortality Table.

At Age.	Number Surviving.	Deaths.	At Age.	Number Surviving.	Deaths.	At Age.	Number Surviving.	Deaths.	At Age.	Number Surviving.	Deaths.
10	100,000	749	35	81,822	732	60	57,917	1,546	85	5,485	1,292
11	99,251	746	36	81,090	737	61	56,371	1,628	86	4,193	1,114
12	98,505	743	37	80,353	742	62	54,743	1,713	87	3,079	933
13	97,762	740	38	79,611	749	63	53,030	1,800	88	2,146	744
14	97,022	737	39	78,862	756	64	51,230	1,889	89	1,402	555
15	96,285	735	40	78,106	765	65	49,341	1,980	90	847	385
16	95,550	732	41	77,341	774	66	47,361	2,070	91	462	246
17	94,818	729	42	76,567	785	67	45,291	2,158	92	216	137
18	94,089	727	43	75,782	797	68	43,133	2,243	93	79	58
19	93,362	725	44	74,985	812	69	40,890	2,321	94	21	18
20	92,637	723	45	74,173	828	70	38,569	2,391	95	3	3
21	91,914	722	46	73,345	848	71	36,178	2,448			
22	91,192	721	47	72,497	870	72	33,730	2,487			
23	90,471	720	48	71,627	896	73	31,243	2,505			
24	89,751	719	49	70,731	927	74	28,738	2,501			
25	89,032	718	50	69,804	962	75	26,237	2,476			
26	88,314	718	51	68,842	1,001	76	23,761	2,431			
27	87,596	718	52	67,841	1,044	77	21,330	2,369			
28	86,878	718	53	66,797	1,091	78	18,961	2,291			
29	86,160	719	54	65,706	1,143	79	16,670	2,196			
30	85,441	720	55	64,563	1,199	80	14,474	2,091			
31	84,721	721	56	63,364	1,260	81	12,383	1,964			
32	84,000	723	57	62,104	1,325	82	10,419	1,816			
33	83,277	726	58	60,779	1,394	83	8,603	1,648			
34	82,551	729	59	59,385	1,468	84	6,955	1,470			

Certain Convenient Values for $n = 1$ to $n = 10$

n	$1/n$	$\sqrt{n}$	$\sqrt[3]{n}$	$n!$	$1/n!$	$\text{Log}_{10} n$
1	1.000000	1.00000	1.00000	1	1.0000000	0.000000000
2	0.500000	1.41421	1.25992	2	0.5000000	0.301029996
3	0.333333	1.73205	1.44225	6	0.1666667	0.477121255
4	0.250000	2.00000	1.58740	24	0.0416667	0.602059991
5	0.200000	2.23607	1.70998	120	0.0083333	0.698970004
6	0.166667	2.44949	1.81712	720	0.0013889	0.778151250
7	0.142857	2.64575	1.91293	5040	0.0001984	0.845098040
8	0.125000	2.82843	2.00000	40320	0.0000248	0.903089987
9	0.111111	3.00000	2.08008	362880	0.0000028	0.954242509
10	0.100000	3.16228	2.15443	3628800	0.0000003	1.000000000

The tables on the following pages are four-place tables of logarithms, trigonometric functions, powers and roots, and compound interest. More extensive tables on all these may be found in The Macmillan Tables, from which these are taken.

TABLE VII. Squares and Cubes, Square Roots, and Cube Roots.

No.	Square	Cube	Square Root	Cube Root	No.	Square	Cube	Square Root	Cube Root
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260	52	2,704	140,608	7.211	3.733
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.780
5	25	125	2.236	1.710	55	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.826
7	49	343	2.646	1.913	57	3,249	185,193	7.550	3.849
8	64	512	2.828	2.000	58	3,364	195,112	7.616	3.871
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	60	3,600	216,000	7.746	3.915
11	121	1,331	3.317	2.224	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.958
13	169	2,197	3.606	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.742	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	65	4,225	274,625	8.062	4.021
16	256	4,096	4.000	2.520	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.062
18	324	5,832	4.243	2.621	68	4,624	314,432	8.246	4.082
19	361	6,859	4.359	2.668	69	4,761	328,509	8.307	4.102
20	400	8,000	4.472	2.714	70	4,900	343,000	8.367	4.121
21	441	9,261	4.583	2.759	71	5,041	357,911	8.426	4.141
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.796	2.844	73	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.718	4.236
27	729	19,683	5.196	3.000	77	5,929	456,533	8.775	4.254
28	784	21,952	5.292	3.037	78	6,084	474,552	8.832	4.273
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.309
31	961	29,791	5.568	3.141	81	6,561	531,441	9.000	4.327
32	1,024	32,768	5.657	3.175	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745	3.208	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240	84	7,056	592,704	9.165	4.380
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.220	4.397
36	1,296	46,656	6.000	3.302	86	7,396	636,056	9.274	4.414
37	1,369	50,653	6.083	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	88	7,744	681,472	9.381	4.448
39	1,521	59,319	6.245	3.391	89	7,921	704,969	9.434	4.465
40	1,600	64,000	6.325	3.420	90	8,100	729,000	9.487	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.498
42	1,764	74,088	6.481	3.476	92	8,464	778,688	9.592	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.644	4.531
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.547
45	2,025	91,125	6.708	3.557	95	9,025	857,375	9.747	4.563
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.798	4.579
47	2,209	103,823	6.856	3.609	97	9,409	912,673	9.849	4.595
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.950	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.642

N	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 8 12	17 21 25	29 33 37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 8 11	15 19 23	26 30 34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 7 10	14 17 21	24 28 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 6 10	13 16 19	23 26 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 6 9	12 15 18	21 24 27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 6 8	11 14 17	20 22 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 5 8	11 13 16	18 21 24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2 5 7	10 12 15	17 20 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 5 7	9 12 14	16 19 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 4 7	9 11 13	16 18 20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2 4 6	8 11 13	15 17 19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2 4 6	8 10 12	14 16 18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2 4 6	8 10 12	14 16 1.
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2 4 6	7 9 11	13 15 17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2 4 5	7 9 11	12 14 16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2 4 5	7 9 10	12 14 15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2 3 5	7 8 10	11 13 15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2 3 5	6 8 9	11 12 14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2 3 5	6 8 9	11 12 14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1 3 4	6 7 9	10 12 13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1 3 4	6 7 9	10 11 13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1 3 4	5 7 8	10 11 12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1 3 4	5 7 8	9 11 12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1 3 4	5 7 8	9 11 12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1 2 4	5 6 8	9 10 11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1 2 4	5 6 7	9 10 11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1 2 4	5 6 7	8 10 11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1 2 4	5 6 7	8 9 11
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1 2 3	5 6 7	8 9 10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1 2 3	4 5 7	8 9 10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1 2 3	4 5 6	8 9 10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1 2 3	4 5 6	7 8 9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1 2 3	4 5 6	7 8 9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1 2 3	4 5 6	7 8 9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1 2 3	4 5 6	7 8 9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1 2 3	4 5 6	7 8 9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1 2 3	4 5 6	7 7 8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1 2 3	4 5 6	7 7 8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1 2 3	4 5 6	7 7 8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1 2 3	4 4 5	6 7 8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 3	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7
N	0	1	2	3	4	5	6	7	8	9	1 2 2	4 5 6	7 8 9

The proportional parts are stated in full for every tenth at the right-hand side. The logarithm of any number of four significant figures can be read directly by add-

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	1	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	3	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	5	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	3	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	3	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	3	3	4	4	5	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	3	3	4	4	5	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	4	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	1	1	2	2	3	3	4	4	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	3	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

ing the proportional part corresponding to the fourth figure to the tabular number corresponding to the first three figures. There may be an error of 1 in the last place.

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.0000	0° 00′	.0000	—	.0000	—	—	—	1.0000	.0000	90° 00′	1.5708
.0029	10	.0029	.4637	.0029	.4637	343.77	.5363	1.0000	.0000	50	1.5679
.0058	20	.0058	.7648	.0058	.7648	171.89	.2352	1.0000	.0000	40	1.5650
.0087	30	.0087	.9408	.0087	.9409	114.59	.0591	1.0000	.0000	30	1.5621
.0116	40	.0116	.0658	.0116	.0658	85.940	.9342	.9999	.0000	20	1.5592
.0145	50	.0145	.1627	.0145	.1627	68.750	.8373	.9999	.0000	10	1.5563
.0175	1° 00′	.0175	.2419	.0175	.2419	57.290	.7581	.9998	.9999	89° 00′	1.5533
.0204	10	.0204	.3088	.0204	.3089	49.104	.6911	.9998	.9999	50	1.5504
.0233	20	.0233	.3668	.0233	.3669	42.964	.6331	.9997	.9999	40	1.5475
.0262	30	.0262	.4179	.0262	.4181	38.188	.5819	.9997	.9999	30	1.5446
.0291	40	.0291	.4637	.0291	.4638	34.368	.5362	.9996	.9998	20	1.5417
.0320	50	.0320	.5050	.0320	.5053	31.242	.4947	.9995	.9998	10	1.5388
.0349	2° 00′	.0349	.5428	.0349	.5431	28.636	.4569	.9994	.9997	88° 00′	1.5359
.0378	10	.0378	.5776	.0378	.5779	26.432	.4221	.9993	.9997	50	1.5330
.0407	20	.0407	.6097	.0407	.6101	24.542	.3899	.9992	.9996	40	1.5301
.0436	30	.0436	.6397	.0437	.6401	22.904	.3599	.9990	.9996	30	1.5272
.0465	40	.0465	.6677	.0466	.6682	21.470	.3318	.9989	.9995	20	1.5243
.0495	50	.0494	.6940	.0495	.6945	20.206	.3055	.9988	.9995	10	1.5213
.0524	3° 00′	.0523	.7188	.0524	.7194	19.081	.2806	.9986	.9994	87° 00′	1.5184
.0553	10	.0552	.7423	.0553	.7429	18.075	.2571	.9985	.9993	50	1.5155
.0582	20	.0581	.7645	.0582	.7652	17.169	.2348	.9983	.9993	40	1.5126
.0611	30	.0610	.7857	.0612	.7865	16.350	.2135	.9981	.9992	30	1.5097
.0640	40	.0640	.8059	.0641	.8067	15.605	.1933	.9980	.9991	20	1.5068
.0669	50	.0669	.8251	.0670	.8261	14.924	.1739	.9978	.9990	10	1.5039
.0698	4° 00′	.0698	.8436	.0699	.8446	14.301	.1554	.9976	.9989	86° 00′	1.5010
.0727	10	.0727	.8613	.0729	.8624	13.727	.1376	.9974	.9989	50	1.4981
.0756	20	.0756	.8783	.0758	.8795	13.197	.1205	.9971	.9988	40	1.4952
.0785	30	.0785	.8946	.0787	.8960	12.706	.1040	.9969	.9987	30	1.4923
.0814	40	.0814	.9104	.0816	.9118	12.251	.0882	.9967	.9986	20	1.4893
.0844	50	.0843	.9256	.0846	.9272	11.826	.0728	.9964	.9985	10	1.4864
.0873	5° 00′	.0872	.9403	.0875	.9420	11.430	.0580	.9962	.9983	85° 00′	1.4835
.0902	10	.0901	.9545	.0904	.9563	11.059	.0437	.9959	.9982	50	1.4806
.0931	20	.0929	.9682	.0934	.9701	10.712	.0299	.9957	.9981	40	1.4777
.0960	30	.0958	.9816	.0963	.9836	10.385	.0164	.9954	.9980	30	1.4748
.0989	40	.0987	.9945	.0992	.9966	10.078	.0034	.9951	.9979	20	1.4719
.1018	50	.1016	.0070	.1022	.0093	9.7882	.9907	.9948	.9977	10	1.4690
.1047	6° 00′	.1045	.0192	.1051	.0216	9.5144	.9784	.9945	.9976	84° 00′	1.4661
.1076	10	.1074	.0311	.1080	.0336	9.2553	.9664	.9942	.9975	50	1.4632
.1105	20	.1103	.0426	.1110	.0453	9.0098	.9547	.9939	.9973	40	1.4603
.1134	30	.1132	.0539	.1139	.0567	8.7769	.9433	.9936	.9972	30	1.4573
.1164	40	.1161	.0648	.1169	.0678	8.5555	.9322	.9932	.9971	20	1.4544
.1193	50	.1190	.0755	.1198	.0786	8.3450	.9214	.9929	.9969	10	1.4515
.1222	7° 00′	.1219	.0859	.1228	.0891	8.1443	.9109	.9925	.9968	83° 00′	1.4486
.1251	10	.1248	.0961	.1257	.0995	7.9530	.9005	.9922	.9966	50	1.4457
.1280	20	.1276	.1060	.1287	.1096	7.7704	.8904	.9918	.9964	40	1.4428
.1309	30	.1305	.1157	.1317	.1194	7.5958	.8806	.9914	.9963	30	1.4399
.1338	40	.1334	.1252	.1346	.1291	7.4287	.8709	.9911	.9961	20	1.4370
.1367	50	.1363	.1345	.1376	.1385	7.2687	.8615	.9907	.9959	10	1.4341
.1396	8° 00′	.1392	.1436	.1405	.1478	7.1154	.8522	.9903	.9958	82° 00′	1.4312
.1425	10	.1421	.1525	.1435	.1569	6.9682	.8431	.9899	.9956	50	1.4283
.1454	20	.1449	.1612	.1465	.1658	6.8269	.8342	.9894	.9954	40	1.4254
.1484	30	.1478	.1697	.1495	.1745	6.6912	.8255	.9890	.9952	30	1.4224
.1513	40	.1507	.1781	.1524	.1831	6.5606	.8169	.9886	.9950	20	1.4195
.1542	50	.1536	.1863	.1554	.1915	6.4348	.8085	.9881	.9948	10	1.4166
.1571	9° 00′	.1564	.1943	.1584	.1997	6.3138	.8003	.9877	.9946	81° 00′	1.4137
		Value	Cosine $\log_{10}$	Value	Cotangent $\log_{10}$	Value	Tangent $\log_{10}$	Value	Sine $\log_{10}$	Degrees	Radians

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.1571	**9° 00′**	.1564	.1943	.1584	.1997	6.3138	.8003	.9877	.9946	**81° 00′**	1.4137
.1600	10	.1593	.2022	.1614	.2078	6.1970	.7922	.9872	.9944	50	1.4108
.1629	20	.1622	.2100	.1644	.2158	6.0844	.7842	.9868	.9942	40	1.4079
.1658	30	.1650	.2176	.1673	.2236	5.9758	.7764	.9863	.9940	30	1.4050
.1687	40	.1679	.2251	.1703	.2313	5.8708	.7687	.9858	.9938	20	1.4021
.1716	50	.1708	.2324	.1733	.2389	5.7694	.7611	.9853	.9936	10	1.3992
.1745	**10° 00′**	.1736	.2397	.1763	.2463	5.6713	.7537	.9848	.9934	**80° 00′**	1.3963
.1774	10	.1765	.2468	.1793	.2536	5.5764	.7464	.9843	.9931	50	1.3934
.1804	20	.1794	.2538	.1823	.2609	5.4845	.7391	.9838	.9929	40	1.3904
.1833	30	.1822	.2606	.1853	.2680	5.3955	.7320	.9833	.9927	30	1.3875
.1862	40	.1851	.2674	.1883	.2750	5.3093	.7250	.9827	.9924	20	1.3846
.1891	50	.1880	.2740	.1914	.2819	5.2257	.7181	.9822	.9922	10	1.3817
.1920	**11° 00′**	.1908	.2806	.1944	.2887	5.1446	.7113	.9816	.9919	**79° 00′**	1.3788
.1949	10	.1937	.2870	.1974	.2953	5.0658	.7047	.9811	.9917	50	1.3759
.1978	20	.1965	.2934	.2004	.3020	4.9894	.6980	.9805	.9914	40	1.3730
.2007	30	.1994	.2997	.2035	.3085	4.9152	.6915	.9799	.9912	30	1.3701
.2036	40	.2022	.3058	.2065	.3149	4.8430	.6851	.9793	.9909	20	1.3672
.2065	50	.2051	.3119	.2095	.3212	4.7729	.6788	.9787	.9907	10	1.3643
.2094	**12° 00′**	.2079	.3179	.2126	.3275	4.7046	.6725	.9781	.9904	**78° 00′**	1.3614
.2123	10	.2108	.3238	.2156	.3336	4.6382	.6664	.9775	.9901	50	1.3584
.2153	20	.2136	.3296	.2186	.3397	4.5736	.6603	.9769	.9899	40	1.3555
.2182	30	.2164	.3353	.2217	.3458	4.5107	.6542	.9763	.9896	30	1.3526
.2211	40	.2193	.3410	.2247	.3517	4.4494	.6483	.9757	.9893	20	1.3497
.2240	50	.2221	.3466	.2278	.3576	4.3897	.6424	.9750	.9890	10	1.3468
.2269	**13° 00′**	.2250	.3521	.2309	.3634	4.3315	.6366	.9744	.9887	**77° 00′**	1.3439
.2298	10	.2278	.3575	.2339	.3691	4.2747	.6309	.9737	.9884	50	1.3410
.2327	20	.2306	.3629	.2370	.3748	4.2193	.6252	.9730	.9881	40	1.3381
.2356	30	.2334	.3682	.2401	.3804	4.1653	.6196	.9724	.9878	30	1.3352
.2385	40	.2363	.3734	.2432	.3859	4.1126	.6141	.9717	.9875	20	1.3323
.2414	50	.2391	.3786	.2462	.3914	4.0611	.6086	.9710	.9872	10	1.3294
.2443	**14° 00′**	.2419	.3837	.2493	.3968	4.0108	.6032	.9703	.9869	**76° 00′**	1.3265
.2473	10	.2447	.3887	.2524	.4021	3.9617	.5979	.9696	.9866	50	1.3235
.2502	20	.2476	.3937	.2555	.4074	3.9136	.5926	.9689	.9863	40	1.3206
.2531	30	.2504	.3986	.2586	.4127	3.8667	.5873	.9681	.9859	30	1.3177
.2560	40	.2532	.4035	.2617	.4178	3.8208	.5822	.9674	.9856	20	1.3148
.2589	50	.2560	.4083	.2648	.4230	3.7760	.5770	.9667	.9853	10	1.3119
.2618	**15° 00′**	.2588	.4130	.2679	.4281	3.7321	.5719	.9659	.9849	**75° 00′**	1.3090
.2647	10	.2616	.4177	.2711	.4331	3.6891	.5669	.9652	.9846	50	1.3061
.2676	20	.2644	.4223	.2742	.4381	3.6470	.5619	.9644	.9843	40	1.3032
.2705	30	.2672	.4269	.2773	.4430	3.6059	.5570	.9636	.9839	30	1.3003
.2734	40	.2700	.4314	.2805	.4479	3.5656	.5521	.9628	.9836	20	1.2974
.2763	50	.2728	.4359	.2836	.4527	3.5261	.5473	.9621	.9832	10	1.2945
.2793	**16° 00′**	.2756	.4403	.2867	.4575	3.4874	.5425	.9613	.9828	**74° 00′**	1.2915
.2822	10	.2784	.4447	.2899	.4622	3.4495	.5378	.9605	.9825	50	1.2886
.2851	20	.2812	.4491	.2931	.4669	3.4124	.5331	.9596	.9821	40	1.2857
.2880	30	.2840	.4533	.2962	.4716	3.3759	.5284	.9588	.9817	30	1.2828
.2909	40	.2868	.4576	.2994	.4762	3.3402	.5238	.9580	.9814	20	1.2799
.2938	50	.2896	.4618	.3026	.4808	3.3052	.5192	.9572	.9810	10	1.2770
.2967	**17° 00′**	.2924	.4659	.3057	.4853	3.2709	.5147	.9563	.9806	**73° 00′**	1.2741
.2996	10	.2952	.4700	.3089	.4898	3.2371	.5102	.9555	.9802	50	1.2712
.3025	20	.2979	.4741	.3121	.4943	3.2041	.5057	.9546	.9798	40	1.2683
.3054	30	.3007	.4781	.3153	.4987	3.1716	.5013	.9537	.9794	30	1.2654
.3083	40	.3035	.4821	.3185	.5031	3.1397	.4969	.9528	.9790	20	1.2625
.3113	50	.3062	.4861	.3217	.5075	3.1084	.4925	.9520	.9786	10	1.2595
.3142	**18° 00′**	.3090	.4900	.3249	.5118	3.0777	.4882	.9511	.9782	**72° 00′**	1.2566
		Value	Cosine $\log_{10}$	Value	Cotangent $\log_{10}$	Value	Tangent $\log_{10}$	Value	Sine $\log_{10}$	Degrees	Radians

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.3142	**18° 00′**	.3090	.4900	.3249	.5118	3.0777	.4882	.9511	.9782	**72° 00′**	1.2566
.3171	10	.3118	.4939	.3281	.5161	3.0475	.4839	.9502	.9778	50	1.2537
.3200	20	.3145	.4977	.3314	.5203	3.0178	.4797	.9492	.9774	40	1.2508
.3229	30	.3173	.5015	.3346	.5245	2.9887	.4755	.9483	.9770	30	1.2479
.3258	40	.3201	.5052	.3378	.5287	2.9600	.4713	.9474	.9765	20	1.2450
.3287	50	.3228	.5090	.3411	.5329	2.9319	.4671	.9465	.9761	10	1.2421
.3316	**19° 00′**	.3256	.5126	.3443	.5370	2.9042	.4630	.9455	.9757	**71° 00′**	1.2392
.3345	10	.3283	.5163	.3476	.5411	2.8770	.4589	.9446	.9752	50	1.2363
.3374	20	.3311	.5199	.3508	.5451	2.8502	.4549	.9436	.9748	40	1.2334
.3403	30	.3338	.5235	.3541	.5491	2.8239	.4509	.9426	.9743	30	1.2305
.3432	40	.3365	.5270	.3574	.5531	2.7980	.4469	.9417	.9739	20	1.2275
.3462	50	.3393	.5306	.3607	.5571	2.7725	.4429	.9407	.9734	10	1.2246
.3491	**20° 00′**	.3420	.5341	.3640	.5611	2.7475	.4389	.9397	.9730	**70° 00′**	1.2217
.3520	10	.3448	.5375	.3673	.5650	2.7228	.4350	.9387	.9725	50	1.2188
.3549	20	.3475	.5409	.3706	.5689	2.6985	.4311	.9377	.9721	40	1.2159
.3578	30	.3502	.5443	.3739	.5727	2.6746	.4273	.9367	.9716	30	1.2130
.3607	40	.3529	.5477	.3772	.5766	2.6511	.4234	.9356	.9711	20	1.2101
.3636	50	.3557	.5510	.3805	.5804	2.6279	.4196	.9346	.9706	10	1.2072
.3665	**21° 00′**	.3584	.5543	.3839	.5842	2.6051	.4158	.9336	.9702	**69° 00′**	1.2043
.3694	10	.3611	.5576	.3872	.5879	2.5826	.4121	.9325	.9697	50	1.2014
.3723	20	.3638	.5609	.3906	.5917	2.5605	.4083	.9315	.9692	40	1.1985
.3752	30	.3665	.5641	.3939	.5954	2.5386	.4046	.9304	.9687	30	1.1956
.3782	40	.3692	.5673	.3973	.5991	2.5172	.4009	.9293	.9682	20	1.1926
.3811	50	.3719	.5704	.4006	.6028	2.4960	.3972	.9283	.9677	10	1.1897
.3840	**22° 00′**	.3746	.5736	.4040	.6064	2.4751	.3936	.9272	.9672	**68° 00′**	1.1868
.3869	10	.3773	.5767	.4074	.6100	2.4545	.3900	.9261	.9667	50	1.1839
.3898	20	.3800	.5798	.4108	.6136	2.4342	.3864	.9250	.9661	40	1.1810
.3927	30	.3827	.5828	.4142	.6172	2.4142	.3828	.9239	.9656	30	1.1781
.3956	40	.3854	.5859	.4176	.6208	2.3945	.3792	.9228	.9651	20	1.1752
.3985	50	.3881	.5889	.4210	.6243	2.3750	.3757	.9216	.9646	10	1.1723
.4014	**23° 00′**	.3907	.5919	.4245	.6279	2.3559	.3721	.9205	.9640	**67° 00′**	1.1694
.4043	10	.3934	.5948	.4279	.6314	2.3369	.3686	.9194	.9635	50	1.1665
.4072	20	.3961	.5978	.4314	.6348	2.3183	.3652	.9182	.9629	40	1.1636
.4102	30	.3987	.6007	.4348	.6383	2.2998	.3617	.9171	.9624	30	1.1606
.4131	40	.4014	.6036	.4383	.6417	2.2817	.3583	.9159	.9618	20	1.1577
.4160	50	.4041	.6065	.4417	.6452	2.2637	.3548	.9147	.9613	10	1.1548
.4189	**24° 00′**	.4067	.6093	.4452	.6486	2.2460	.3514	.9135	.9607	**66° 00′**	1.1519
.4218	10	.4094	.6121	.4487	.6520	2.2286	.3480	.9124	.9602	50	1.1490
.4247	20	.4120	.6149	.4522	.6553	2.2113	.3447	.9112	.9596	40	1.1461
.4276	30	.4147	.6177	.4557	.6587	2.1943	.3413	.9100	.9590	30	1.1432
.4305	40	.4173	.6205	.4592	.6620	2.1775	.3380	.9088	.9584	20	1.1403
.4334	50	.4200	.6232	.4628	.6654	2.1609	.3346	.9075	.9579	10	1.1374
.4363	**25° 00′**	.4226	.6259	.4663	.6687	2.1445	.3313	.9063	.9573	**65° 00′**	1.1345
.4392	10	.4253	.6286	.4699	.6720	2.1283	.3280	.9051	.9567	50	1.1316
.4422	20	.4279	.6313	.4734	.6752	2.1123	.3248	.9038	.9561	40	1.1286
.4451	30	.4305	.6340	.4770	.6785	2.0965	.3215	.9026	.9555	30	1.1257
.4480	40	.4331	.6366	.4806	.6817	2.0809	.3183	.9013	.9549	20	1.1228
.4509	50	.4358	.6392	.4841	.6850	2.0655	.3150	.9001	.9543	10	1.1199
.4538	**26° 00′**	.4384	.6418	.4877	.6882	2.0503	.3118	.8988	.9537	**64° 00′**	1.1170
.4567	10	.4410	.6444	.4913	.6914	2.0353	.3086	.8975	.9530	50	1.1141
.4596	20	.4436	.6470	.4950	.6946	2.0204	.3054	.8962	.9524	40	1.1112
.4625	30	.4462	.6495	.4986	.6977	2.0057	.3023	.8949	.9518	30	1.1083
.4654	40	.4488	.6521	.5022	.7009	1.9912	.2991	.8936	.9512	20	1.1054
.4683	50	.4514	.6546	.5059	.7040	1.9768	.2960	.8923	.9505	10	1.1025
.4712	**27° 00′**	.4540	.6570	.5095	.7072	1.9626	.2928	.8910	.9499	**63° 00′**	1.0996
		Value Cosine	$\log_{10}$	Value Cotangent	$\log_{10}$	Value Tangent	$\log_{10}$	Value Sine	$\log_{10}$	Degrees	Radians

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.4712	**27° 00′**	.4540	.6570	.5095	.7072	1.9626	.2928	.8910	.9499	**63° 00′**	1.0996
.4741	10	.4566	.6595	.5132	.7103	1.9486	.2897	.8897	.9492	50	1.0966
.4771	20	.4592	.6620	.5169	.7134	1.9347	.2866	.8884	.9486	40	1.0937
.4800	30	.4617	.6644	.5206	.7165	1.9210	.2835	.8870	.9479	30	1.0908
.4829	40	.4643	.6668	.5243	.7196	1.9074	.2804	.8857	.9473	20	1.0879
.4858	50	.4669	.6692	.5280	.7226	1.8940	.2774	.8843	.9466	10	1.0850
.4887	**28° 00′**	.4695	.6716	.5317	.7257	1.8807	.2743	.8829	.9459	**62° 00′**	1.0821
.4916	10	.4720	.6740	.5354	.7287	1.8676	.2713	.8816	.9453	50	1.0792
.4945	20	.4746	.6763	.5392	.7317	1.8546	.2683	.8802	.9446	40	1.0763
.4974	30	.4772	.6787	.5430	.7348	1.8418	.2652	.8788	.9439	30	1.0734
.5003	40	.4797	.6810	.5467	.7378	1.8291	.2622	.8774	.9432	20	1.0705
.5032	50	.4823	.6833	.5505	.7408	1.8165	.2592	.8760	.9425	10	1.0676
.5061	**29° 00′**	.4848	.6856	.5543	.7438	1.8040	.2562	.8746	.9418	**61° 00′**	1.0647
.5091	10	.4874	.6878	.5581	.7467	1.7917	.2533	.8732	.9411	50	1.0617
.5120	20	.4899	.6901	.5619	.7497	1.7796	.2503	.8718	.9404	40	1.0588
.5149	30	.4924	.6923	.5658	.7526	1.7675	.2474	.8704	.9397	30	1.0559
.5178	40	.4950	.6946	.5696	.7556	1.7556	.2444	.8689	.9390	20	1.0530
.5207	50	.4975	.6968	.5735	.7585	1.7437	.2415	.8675	.9383	10	1.0501
.5236	**30° 00′**	.5000	.6990	.5774	.7614	1.7321	.2386	.8660	.9375	**60° 00′**	1.0472
.5265	10	.5025	.7012	.5812	.7644	1.7205	.2356	.8646	.9368	50	1.0443
.5294	20	.5050	.7033	.5851	.7673	1.7090	.2327	.8631	.9361	40	1.0414
.5323	30	.5075	.7055	.5890	.7701	1.6977	.2299	.8616	.9353	30	1.0385
.5352	40	.5100	.7076	.5930	.7730	1.6864	.2270	.8601	.9346	20	1.0356
.5381	50	.5125	.7097	.5969	.7759	1.6753	.2241	.8587	.9338	10	1.0327
.5411	**31° 00′**	.5150	.7118	.6009	.7788	1.6643	.2212	.8572	.9331	**59° 00′**	1.0297
.5440	10	.5175	.7139	.6048	.7816	1.6534	.2184	.8557	.9323	50	1.0268
.5469	20	.5200	.7160	.6088	.7845	1.6426	.2155	.8542	.9315	40	1.0239
.5498	30	.5225	.7181	.6128	.7873	1.6319	.2127	.8526	.9308	30	1.0210
.5527	40	.5250	.7201	.6168	.7902	1.6212	.2098	.8511	.9300	20	1.0181
.5556	50	.5275	.7222	.6208	.7930	1.6107	.2070	.8496	.9292	10	1.0152
.5585	**32° 00′**	.5299	.7242	.6249	.7958	1.6003	.2042	.8480	.9284	**58° 00′**	1.0123
.5614	10	.5324	.7262	.6289	.7986	1.5900	.2014	.8465	.9276	50	1.0094
.5643	20	.5348	.7282	.6330	.8014	1.5798	.1986	.8450	.9268	40	1.0065
.5672	30	.5373	.7302	.6371	.8042	1.5697	.1958	.8434	.9260	30	1.0036
.5701	40	.5398	.7322	.6412	.8070	1.5597	.1930	.8418	.9252	20	1.0007
.5730	50	.5422	.7342	.6453	.8097	1.5497	.1903	.8403	.9244	10	.9977
.5760	**33° 00′**	.5446	.7361	.6494	.8125	1.5399	.1875	.8387	.9236	**57° 00′**	.9948
.5789	10	.5471	.7380	.6536	.8153	1.5301	.1847	.8371	.9228	50	.9919
.5818	20	.5495	.7400	.6577	.8180	1.5204	.1820	.8355	.9219	40	.9890
.5847	30	.5519	.7419	.6619	.8208	1.5108	.1792	.8339	.9211	30	.9861
.5876	40	.5544	.7438	.6661	.8235	1.5013	.1765	.8323	.9203	20	.9832
.5905	50	.5568	.7457	.6703	.8263	1.4919	.1737	.8307	.9194	10	.9803
.5934	**34° 00′**	.5592	.7476	.6745	.8290	1.4826	.1710	.8290	.9186	**56° 00′**	.9774
.5963	10	.5616	.7494	.6787	.8317	1.4733	.1683	.8274	.9177	50	.9745
.5992	20	.5640	.7513	.6830	.8344	1.4641	.1656	.8258	.9169	40	.9716
.6021	30	.5664	.7531	.6873	.8371	1.4550	.1629	.8241	.9160	30	.9687
.6050	40	.5688	.7550	.6916	.8398	1.4460	.1602	.8225	.9151	20	.9657
.6080	50	.5712	.7568	.6959	.8425	1.4370	.1575	.8208	.9142	10	.9628
.6109	**35° 00′**	.5736	.7586	.7002	.8452	1.4281	.1548	.8192	.9134	**55° 00′**	.9599
.6138	10	.5760	.7604	.7046	.8479	1.4193	.1521	.8175	.9125	50	.9570
.6167	20	.5783	.7622	.7089	.8506	1.4106	.1494	.8158	.9116	40	.9541
.6196	30	.5807	.7640	.7133	.8533	1.4019	.1467	.8141	.9107	30	.9512
.6225	40	.5831	.7657	.7177	.8559	1.3934	.1441	.8124	.9098	20	.9483
.6254	50	.5854	.7675	.7221	.8586	1.3848	.1414	.8107	.9089	10	.9454
.6283	**36° 00′**	.5878	.7692	.7265	.8613	1.3764	.1387	.8090	.9080	**54° 00′**	.9425
		Value	Cosine $\log_{10}$	Value	Cotangent $\log_{10}$	Value	Tangent $\log_{10}$	Value	Sine $\log_{10}$	Degrees	Radians

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.6283	**36° 00′**	.5878	.7692	.7265	.8613	1.3764	.1387	.8090	.9080	**54° 00′**	.9425
.6312	10	.5901	.7710	.7310	.8639	1.3680	.1361	.8073	.9070	50	.9396
.6341	20	.5925	.7727	.7355	.8666	1.3597	.1334	.8056	.9061	40	.9367
.6370	30	.5948	.7744	.7400	.8692	1.3514	.1308	.8039	.9052	30	.9338
.6400	40	.5972	.7761	.7445	.8718	1.3432	.1282	.8021	.9042	20	.9308
.6429	50	.5995	.7778	.7490	.8745	1.3351	.1255	.8004	.9033	10	.9279
.6458	**37° 00′**	.6018	.7795	.7536	.8771	1.3270	.1229	.7986	.9023	**53° 00′**	.9250
.6487	10	.6041	.7811	.7581	.8797	1.3190	.1203	.7969	.9014	50	.9221
.6516	20	.6065	.7828	.7627	.8824	1.3111	.1176	.7951	.9004	40	.9192
.6545	30	.6088	.7844	.7673	.8850	1.3032	.1150	.7934	.8995	30	.9163
.6574	40	.6111	.7861	.7720	.8876	1.2954	.1124	.7916	.8985	20	.9134
.6603	50	.6134	.7877	.7766	.8902	1.2876	.1098	.7898	.8975	10	.9105
.6632	**38° 00′**	.6157	.7893	.7813	.8928	1.2799	.1072	.7880	.8965	**52° 00′**	.9076
.6661	10	.6180	.7910	.7860	.8954	1.2723	.1046	.7862	.8955	50	.9047
.6690	20	.6202	.7926	.7907	.8980	1.2647	.1020	.7844	.8945	40	.9018
.6720	30	.6225	.7941	.7954	.9006	1.2572	.0994	.7826	.8935	30	.8988
.6749	40	.6248	.7957	.8002	.9032	1.2497	.0968	.7808	.8925	20	.8959
.6778	50	.6271	.7973	.8050	.9058	1.2423	.0942	.7790	.8915	10	.8930
.6807	**39° 00′**	.6293	.7989	.8098	.9084	1.2349	.0916	.7771	.8905	**51° 00′**	.8901
.6836	10	.6316	.8004	.8146	.9110	1.2276	.0890	.7753	.8895	50	.8872
.6865	20	.6338	.8020	.8195	.9135	1.2203	.0865	.7735	.8884	40	.8843
.6894	30	.6361	.8035	.8243	.9161	1.2131	.0839	.7716	.8874	30	.8814
.6923	40	.6383	.8050	.8292	.9187	1.2059	.0813	.7698	.8864	20	.8785
.6952	50	.6406	.8066	.8342	.9212	1.1988	.0788	.7679	.8853	10	.8756
.6981	**40° 00′**	.6428	.8081	.8391	.9238	1.1918	.0762	.7660	.8843	**50° 00′**	.8727
.7010	10	.6450	.8096	.8441	.9264	1.1847	.0736	.7642	.8832	50	.8698
.7039	20	.6472	.8111	.8491	.9289	1.1778	.0711	.7623	.8821	40	.8668
.7069	30	.6494	.8125	.8541	.9315	1.1708	.0685	.7604	.8810	30	.8639
.7098	40	.6517	.8140	.8591	.9341	1.1640	.0659	.7585	.8800	20	.8610
.7127	50	.6539	.8155	.8642	.9366	1.1571	.0634	.7566	.8789	10	.8581
.7156	**41° 00′**	.6561	.8169	.8693	.9392	1.1504	.0608	.7547	.8778	**49° 00′**	.8552
.7185	10	.6583	.8184	.8744	.9417	1.1436	.0583	.7528	.8767	50	.8523
.7214	20	.6604	.8198	.8796	.9443	1.1369	.0557	.7509	.8756	40	.8494
.7243	30	.6626	.8213	.8847	.9468	1.1303	.0532	.7490	.8745	30	.8465
.7272	40	.6648	.8227	.8899	.9494	1.1237	.0506	.7470	.8733	20	.8436
.7301	50	.6670	.8241	.8952	.9519	1.1171	.0481	.7451	.8722	10	.8407
.7330	**42° 00′**	.6691	.8255	.9004	.9544	1.1106	.0456	.7431	.8711	**48° 00′**	.8378
.7359	10	.6713	.8269	.9057	.9570	1.1041	.0430	.7412	.8699	50	.8348
.7389	20	.6734	.8283	.9110	.9595	1.0977	.0405	.7392	.8688	40	.8319
.7418	30	.6756	.8297	.9163	.9621	1.0913	.0379	.7373	.8676	30	.8290
.7447	40	.6777	.8311	.9217	.9646	1.0850	.0354	.7353	.8665	20	.8261
.7476	50	.6799	.8324	.9271	.9671	1.0786	.0329	.7333	.8653	10	.8232
.7505	**43° 00′**	.6820	.8338	.9325	.9697	1.0724	.0303	.7314	.8641	**47° 00′**	.8203
.7534	10	.6841	.8351	.9380	.9722	1.0661	.0278	.7294	.8629	50	.8174
.7563	20	.6862	.8365	.9435	.9747	1.0599	.0253	.7274	.8618	40	.8145
.7592	30	.6884	.8378	.9490	.9772	1.0538	.0228	.7254	.8606	30	.8116
.7621	40	.6905	.8391	.9545	.9798	1.0477	.0202	.7234	.8594	20	.8087
.7650	50	.6926	.8405	.9601	.9823	1.0416	.0177	.7214	.8582	10	.8058
.7679	**44° 00′**	.6947	.8418	.9657	.9848	1.0355	.0152	.7193	.8569	**46° 00′**	.8029
.7709	10	.6967	.8431	.9713	.9874	1.0295	.0126	.7173	.8557	50	.7999
.7738	20	.6988	.8444	.9770	.9899	1.0235	.0101	.7153	.8545	40	.7970
.7767	30	.7009	.8457	.9827	.9924	1.0176	.0076	.7133	.8532	30	.7941
.7796	40	.7030	.8469	.9884	.9949	1.0117	.0051	.7112	.8520	20	.7912
.7825	50	.7050	.8482	.9942	.9975	1.0058	.0025	.7092	.8507	10	.7883
.7854	**45° 00′**	.7071	.8495	1.0000	.0000	1.0000	.0000	.7071	.8495	**45° 00′**	.7854
		Value	Cosine $\log_{10}$	Value	Cotangent $\log_{10}$	Value	Tangent $\log_{10}$	Value	Sine $\log_{10}$	Degrees	Radians

AMOUNT OF ONE DOLLAR PRINCIPAL AT COMPOUND INTEREST AFTER n YEARS

n	2 %	2½ %	3 %	3½ %	4 %	4½ %	5 %	6 %	7 %
1	1.0200	1.0250	1.0300	1.0350	1.0400	1.0450	1.0500	1.0600	1.0700
2	1.0404	1.0506	1.0609	1.0712	1.0816	1.0920	1.1025	1.1236	1.1449
3	1.0612	1.0769	1.0927	1.1087	1.1249	1.1412	1.1576	1.1910	1.2250
4	1.0824	1.1038	1.1255	1.1475	1.1699	1.1925	1.2155	1.2625	1.3108
5	1.1041	1.1314	1.1593	1.1877	1.2167	1.2462	1.2763	1.3382	1.4026
6	1.1262	1.1597	1.1941	1.2293	1.2653	1.3023	1.3401	1.4185	1.5007
7	1.1487	1.1887	1.2299	1.2723	1.3159	1.3609	1.4071	1.5036	1.6058
8	1.1717	1.2184	1.2668	1.3168	1.3686	1.4221	1.4775	1.5938	1.7182
9	1.1951	1.2489	1.3048	1.3629	1.4233	1.4861	1.5513	1.6895	1.8385
10	1.2190	1.2801	1.3439	1.4106	1.4802	1.5530	1.6289	1.7908	1.9672
11	1.2434	1.3121	1.3842	1.4600	1.5395	1.6229	1.7103	1.8983	2.1049
12	1.2682	1.3449	1.4258	1.5111	1.6010	1.6959	1.7959	2.0122	2.2522
13	1.2936	1.3785	1.4685	1.5640	1.6651	1.7722	1.8856	2.1329	2.4098
14	1.3195	1.4130	1.5126	1.6187	1.7317	1.8519	1.9799	2.2609	2.5785
15	1.3459	1.4483	1.5580	1.6753	1.8009	1.9353	2.0789	2.3966	2.7590
16	1.3728	1.4845	1.6047	1.7340	1.8730	2.0224	2.1829	2.5404	2.9522
17	1.4002	1.5216	1.6528	1.7947	1.9479	2.1134	2.2920	2.6928	3.1588
18	1.4282	1.5597	1.7024	1.8575	2.0258	2.2085	2.4066	2.8543	3.3799
19	1.4568	1.5987	1.7535	1.9225	2.1068	2.3079	2.5270	3.0256	3.6165
20	1.4859	1.6386	1.8061	1.9898	2.1911	2.4117	2.6533	3.2071	3.8697
21	1.5157	1.6796	1.8603	2.0594	2.2788	2.5202	2.7860	3.3996	4.1406
22	1.5460	1.7216	1.9161	2.1315	2.3699	2.6337	2.9253	3.6035	4.4304
23	1.5769	1.7646	1.9736	2.2061	2.4647	2.7522	3.0715	3.8197	4.7405
24	1.6084	1.8087	2.0328	2.2833	2.5633	2.8760	3.2251	4.0489	5.0724
25	1.6406	1.8539	2.0938	2.3632	2.6658	3.0054	3.3864	4.2919	5.4274
26	1.6734	1.9003	2.1566	2.4460	2.7725	3.1407	3.5557	4.5494	5.8074
27	1.7069	1.9478	2.2213	2.5316	2.8834	3.2820	3.7335	4.8223	6.2139
28	1.7410	1.9965	2.2879	2.6202	2.9987	3.4297	3.9201	5.1117	6.6488
29	1.7758	2.0464	2.3566	2.7119	3.1187	3.5840	4.1161	5.4184	7.1143
30	1.8114	2.0976	2.4273	2.8068	3.2434	3.7453	4.3219	5.7435	7.6123
31	1.8476	2.1500	2.5001	2.9050	3.3731	3.9139	4.5380	6.0881	8.1451
32	1.8845	2.2038	2.5751	3.0067	3.5081	4.0900	4.7649	6.4534	8.7153
33	1.9222	2.2589	2.6523	3.1119	3.6484	4.2740	5.0032	6.8406	9.3253
34	1.9607	2.3153	2.7319	3.2209	3.7943	4.4664	5.2533	7.2510	9.9781
35	1.9999	2.3732	2.8139	3.3336	3.9461	4.6673	5.5160	7.6861	10.6766
36	2.0399	2.4325	2.8983	3.4503	4.1039	4.8774	5.7918	8.1473	11.4239
37	2.0807	2.4933	2.9852	3.5710	4.2681	5.0969	6.0814	8.6361	12.2236
38	2.1223	2.5557	3.0748	3.6960	4.4388	5.3262	6.3855	9.1543	13.0793
39	2.1647	2.6196	3.1670	3.8254	4.6164	5.5659	6.7048	9.7035	13.9948
40	2.2080	2.6851	3.2620	3.9593	4.8010	5.8164	7.0400	10.2857	14.9745
41	2.2522	2.7522	3.3599	4.0978	4.9931	6.0781	7.3920	10.9029	16.0227
42	2.2972	2.8210	3.4607	4.2413	5.1928	6.3516	7.7616	11.5570	17.1443
43	2.3432	2.8915	3.5645	4.3897	5.4005	6.6374	8.1497	12.2505	18.3444
44	2.3901	2.9638	3.6715	4.5433	5.6165	6.9361	8.5572	12.9855	19.6285
45	2.4379	3.0379	3.7816	4.7024	5.8412	7.2482	8.9850	13.7646	21.0025
46	2.4866	3.1139	3.8950	4.8669	6.0748	7.5744	9.4343	14.5905	22.4726
47	2.5363	3.1917	4.0119	5.0373	6.3178	7.9153	9.9060	15.4659	24.0457
48	2.5871	3.2715	4.1323	5.2136	6.5705	8.2715	10.4013	16.3939	25.7289
49	2.6388	3.3533	4.2562	5.3961	6.8333	8.6437	10.9213	17.3775	27.5299
50	2.6916	3.4371	4.3839	5.5849	7.1067	9.0326	11.4674	18.4202	29.4570

Table XI — Compound Discount: $1/(1+r)^n$

Present Value of One Dollar Due at the End of n Years

n	2 %	2½ %	3 %	3½ %	4 %	4½ %	5 %	6 %	7 %
1	.98039	.97561	.97087	.96618	.96154	.95694	.95238	.94340	.93458
2	.96117	.95181	.94260	.93351	.92456	.91573	.90703	.89000	.87344
3	.94232	.92860	.91514	.90194	.88900	.87630	.86384	.83962	.81630
4	.92385	.90595	.88849	.87144	.85480	.83856	.82270	.79209	.76290
5	.90573	.88385	.86261	.84197	.82193	.80245	.78353	.74726	.71299
6	.88797	.86230	.83748	.81350	.79031	.76790	.74622	.70496	.66634
7	.87056	.84127	.81309	.78599	.75992	.73483	.71068	.66506	.62275
8	.85349	.82075	.78941	.75941	.73069	.70319	.67684	.62741	.58201
9	.83676	.80073	.76642	.73373	.70259	.67290	.64461	.59190	.54393
10	.82035	.78120	.74409	.70892	.67556	.64393	.61391	.55839	.50835
11	.80426	.76214	.72242	.68495	.64958	.61620	.58468	.52679	.47509
12	.78849	.74356	.70138	.66178	.62460	.58966	.55684	.49697	.44401
13	.77303	.72542	.68095	.63940	.60057	.56427	.53032	.46884	.41496
14	.75788	.70773	.66112	.61778	.57748	.53997	.50507	.44230	.38782
15	.74301	.69047	.64186	.59689	.55526	.51672	.48102	.41727	.36245
16	.72845	.67362	.62317	.57671	.53391	.49447	.45811	.39365	.33873
17	.71416	.65720	.60502	.55720	.51337	.47318	.43630	.37136	.31657
18	.70016	.64117	.58739	.53836	.49363	.45280	.41552	.35034	.29586
19	.68643	.62553	.57029	.52016	.47464	.43330	.39573	.33051	.27651
20	.67297	.61027	.55368	.50257	.45639	.41464	.37689	.31180	.25842
21	.65978	.59539	.53755	.48557	.43883	.39679	.35894	.29416	.24151
22	.64684	.58086	.52189	.46915	.42196	.37970	.34185	.27751	.22571
23	.63416	.56670	.50669	.45329	.40573	.36335	.32557	.26180	.21095
24	.62172	.55288	.49193	.43796	.39012	.34770	.31007	.24698	.19715
25	.60953	.53939	.47761	.42315	.37512	.33273	.29530	.23300	.18425
26	.59758	.52623	.46369	.40884	.36069	.31840	.28124	.21981	.17220
27	.58586	.51340	.45019	.39501	.34682	.30469	.26785	.20737	.16093
28	.57437	.50088	.43708	.38165	.33348	.29157	.25509	.19563	.15040
29	.56311	.48866	.42435	.36875	.32065	.27902	.24295	.18456	.14056
30	.55207	.47674	.41199	.35628	.30832	.26700	.23138	.17411	.13137
31	.54125	.46511	.39999	.34423	.29646	.25550	.22036	.16425	.12277
32	.53063	.45377	.38834	.33259	.28506	.24450	.20987	.15496	.11474
33	.52023	.44270	.37703	.32134	.27409	.23397	.19987	.14619	.10723
34	.51003	.43191	.36604	.31048	.26355	.22390	.19035	.13791	.10022
35	.50003	.42137	.35538	.29998	.25342	.21425	.18129	.13011	.09366
36	.49022	.41109	.34503	.28983	.24367	.20503	.17266	.12274	.08754
37	.48061	.40107	.33498	.28003	.23430	.19620	.16444	.11580	.08181
38	.47119	.39128	.32523	.27056	.22529	.18775	.15661	.10924	.07646
39	.46195	.38174	.31575	.26141	.21662	.17967	.14915	.10306	.07146
40	.45289	.37243	.30656	.25257	.20829	.17193	.14205	.09722	.06678
41	.44401	.36335	.29763	.24403	.20028	.16453	.13528	.09172	.06241
42	.43530	.35448	.28896	.23578	.19257	.15744	.12884	.08653	.05833
43	.42677	.34584	.28054	.22781	.18517	.15066	.12270	.08163	.05451
44	.41840	.33740	.27237	.22010	.17805	.14417	.11686	.07701	.05095
45	.41020	.32917	.26444	.21266	.17120	.13796	.11130	.07265	.04761
46	.40215	.32115	.25674	.20547	.16461	.13202	.10600	.06854	.04450
47	.39427	.31331	.24926	.19852	.15828	.12634	.10095	.06466	.04159
48	.38654	.30567	.24200	.19181	.15219	.12090	.09614	.06100	.03887
49	.37896	.29822	.23495	.18532	.14634	.11569	.09156	.05755	.03632
50	.37153	.29094	.22811	.17905	.14071	.11071	.08720	.05429	.03395

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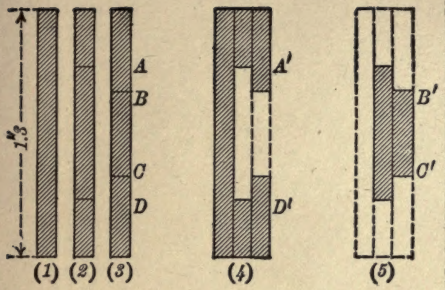
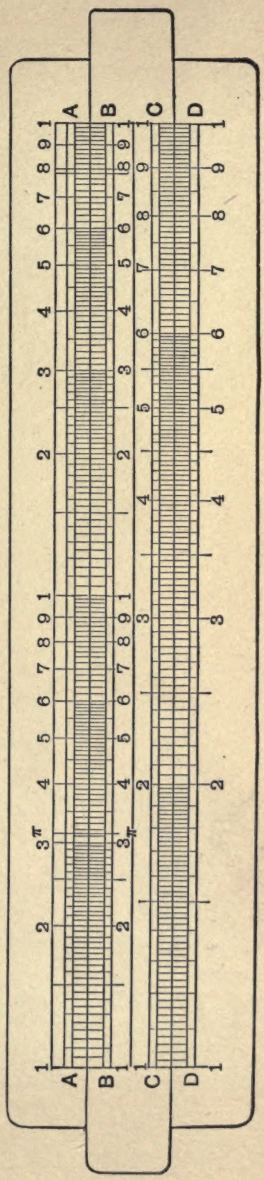


Directions

A reasonably accurate slide rule may be made by the student, for temporary practice, as follows. Take three strips of heavy, stiff cardboard 1″.3 wide by 6″ long; these are shown in cross-section in (1), (2), (3) above. On (3) paste or glue the adjoining cut of the slide rule. Then cut strips (2) and (3) accurately along the lines marked. Paste or glue the pieces together as shown in (4) and (5). Then (5) forms the slide of the slide rule, and it will fit in the groove in (4) if the work has been carefully done. Trim off the ends as shown in the large cut.



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